

The k th Smallest Dirac Operator Eigenvalue and the Pion Decay Constant

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Abstract

We derive an analytical expression for the distribution of the k th smallest Dirac eigenvalue in QCD with imaginary isospin chemical potential in the Dirac operator for arbitrary gauge field topology ν . Because of its dependence on the pion decay constant F_π through the chemical potential in the epsilon-regime of chiral perturbation theory this can be used for lattice determinations of that low-energy constant. On the technical side we use a chiral Random-Two Matrix Theory, where we express the k th eigenvalue distribution through the joint probability of the ordered k smallest eigenvalues. The latter can be computed exactly for finite and infinite N , for which we derive generalisations of Dyson's integration Theorem and Sonine's identity.

1 Introduction

It is by now well known how the spontaneous breaking of chiral symmetry in QCD leads to remarkably strong predictions for the spectral properties of the Dirac operator in that theory. Based first exclusively on the relation to the effective field theory for the associated Nambu-Goldstone bosons at fixed gauge field topology [1], an intriguing relation to universal Random Matrix Theory (RMT) was also pointed out [2]. It has subsequently become clear how these two alternative formulations are related, and all n -point spectral correlation functions have been shown to be identical in these two formulations to leading order in a $1/L$ -expansion (where $L \equiv V^{1/4}$ gives the extent of the space-time volume V) [3, 4]. This holds then also for individual distributions of Dirac operator eigenvalues [5].

If one seeks sensitivity to the pion decay constant F_π it turns out to be useful to consider the Dirac operator of quark doublets with isospin chemical potential μ . Based on the chiral Lagrangian formulation [6], it has been suggested to use a spectral 2-point function of the two associated Dirac operators with imaginary isospin chemical potential. The advantage of imaginary chemical potential lies in the fact that the corresponding Dirac operator retains its anti-hermiticity. To leading order, all results can be expressed in terms of the simple finite-volume scaling variable $\hat{\mu} = \mu F_\pi \sqrt{V}$. In this way, F_π can be extracted from fits that vary μ and/or V . There is also sensitivity to F_π in other observables that couple to chemical potential [7, 8, 9, 10, 11].

The leading-order chiral Lagrangian computations of ref. [6] have been given a reformulation in terms of a chiral Random Two-Matrix Theory in ref. [12]. In this way, all spectral correlation functions associated with the two Dirac operators \mathcal{D}_1 and \mathcal{D}_2 , with respective chemical potentials μ_1 and μ_2 , have been computed analytically in [12] for both the quenched and the full theory with N_f light flavours. It also includes all spectral correlation functions where the imaginary isospin chemical potential only enters in the Dirac operator whose eigenvalues are being computed, while the gauge field configurations are obtained in the usual way at vanishing chemical potential. In analogy with what is being done when varying quark masses away from the value used for generating the gauge field configurations we call this “partial quenching”.

In an earlier paper [13] it was shown how all probability distributions of individual Dirac operator eigenvalues can be computed by means of a series expansion in higher n -point spectral correlation functions. In reference [13] an explicit analytical formula was also given for the lowest non-zero eigenvalue distribution for arbitrary combinations of flavours N_1 and N_2 of the Dirac operators \mathcal{D}_1 and \mathcal{D}_2 , respectively, at gauge field topology $\nu = 0$, in an approach quite close to that of ref. [14]. Two obvious questions remained open that we will answer in this paper: how to extend this to non-zero topology $\nu > 0$, and how to compute the distribution of the second, third or general k th eigenvalue as these are known for zero chemical potential [15]. However, the path of ref. [13] is not very suitable in particular for the derivation of the distributions of higher eigenvalues in a compact analytical manner. Our setup will follow closely ref. [15] at vanishing chemical potential. We shall present here a new formalism that immediately allows for the analytical determination of the distributions of these higher eigenvalues. The extension of the first approach [13] to $\nu > 0$ for the first eigenvalue is presented in appendix C, as an alternative formulation and analytical check to part of our new approach. The benefit of our new results should be two-fold: while expressions for higher topology $\nu > 0$ allow for an independent determination of Σ and F_π from different lattice configurations, the expressions for higher eigenvalues should allow for a better determination using the same configurations as for the first eigenvalue.

How much of the program proposed in [6] to determine F_π on the lattice has been realised in the meantime? Based on a preliminary account [16] of ref. [13], the expansion of the first eigenvalue was first used in simulations in [17]. However, the question remained how large the finite-volume corrections to the leading order (LO) ϵ -expansion are, in which the chiral Lagrangian-RMT correspondence holds.

In a series of papers this question has been addressed and answered: in [18] the next to LO corrections (NLO) and in [19] next to next to LO (NNLO) corrections in the epsilon-expansion were computed. As a result of these computations at NLO all RMT expressions for arbitrary n -point density correlations functions (and thus for all individual eigenvalues too) remain valid. The infinite volume expressions simply get renormalised by finite-volume corrections, one only has to replace Σ and F_π by Σ_{eff} and $F_{\pi eff}$ in the corresponding RMT expressions. Here the subscript “*eff*” for effective encodes the corrections that match those computed earlier in [20] and [21, 11], respectively. Only at NNLO non-universal, non-RMT corrections appear. It was further noticed in [18, 19], that the size of the corrections at each order depends considerably on the lattice geometry, in particular when using asymmetric geometries. In order to keep the NNLO corrections small, in [22] the authors used a specific optimised geometry where they could apply RMT predictions and effective couplings at NLO only, and they obtained realistic values for Σ and F_π from partially quenched lattice data for small chemical potential. Technically speaking Σ was determined there from the first eigenvalue distribution at vanishing chemical potential, and F_π from the shift of the eigenvalues compared to zero chemical potential. We refer to [22] for a more detailed discussion of these fits.

Motivated by these findings we have completed the computation for the k th Dirac eigenvalues for all $\nu \geq 0$ in the RMT setting, in order to have a more complete mathematical toolbox at hand.

Our paper is organised as follows. In the next section we briefly define the notation and remind the reader of the definition of chiral Random Two-Matrix Theory. We introduce a certain joint probability density and describe how it can be used to derive individual eigenvalue distributions. We give the explicit finite- N solution here in terms of new polynomials and a new sequence of matrix model kernels. We take the scaling limit relevant for QCD in section 4, and write out explicitly and discuss the physically most important examples such as partially quenched $N_f = 2$ results. Section 5 contains our conclusions and a suggestion for a quite non-trivial but important extension of these results. Because some of the relevant technical details have been described in ref. [13], we have relegated many of the mathematical details in this paper to appendices. In addition, in Appendix A we describe an explicit construction of the polynomials needed to compute the first eigenvalue distribution in sectors of non-trivial gauge field topology if one alternatively uses the method of ref. [13].

2 Chiral Random Two-Matrix Theory

Before turning to the relevant Random Two-Matrix Theory, we first briefly outline the set-up in the language of the gauge field theory. We are considering QCD at finite four-volume V , and we assume that chiral symmetry is spontaneously broken at infinite volume. We consider two Dirac operators $D_{1,2}$ with different imaginary baryon (quark) chemical potential $\mu_{1,2}$,

$$\begin{aligned} D_1 \psi_1^{(n)} &\equiv [\not{D}(A) + i\mu_1 \gamma_0] \psi_1^{(n)} = i\lambda_1^{(n)} \psi_1^{(n)} \\ D_2 \psi_2^{(n)} &\equiv [\not{D}(A) + i\mu_2 \gamma_0] \psi_2^{(n)} = i\lambda_2^{(n)} \psi_2^{(n)}. \end{aligned} \quad (2.1)$$

When $\mu \equiv \mu_1 = -\mu_2$ this is simply imaginary isospin chemical potential, but we can stay with the more general case. We thus consider N_1 light quarks coupled to quark chemical potential μ_1 , and N_2 light quarks coupled to quark chemical potential μ_2 . Let us first consider the conceptually simplest case where $N_1 + N_2 = N_f$, and later comment on the changes needed to deal with partial quenching.

In the chiral Lagrangian framework the terms that depend on $\mu_{1,2}$ are easily written down on the basis of the usual correspondence with external vector sources. Going to the ϵ -regime of chiral perturbation theory in sectors of fixed gauge field topology [1], the leading term in the effective

partition function including imaginary $\mu_{1,2}$ reads [6, 23] (see also [24] for QCD-like theories)

$$Z_\nu^{(N_f)} = \int_{U(N_f)} dU (\det U)^\nu e^{\frac{1}{4}V F_\pi^2 \text{Tr}[U, B][U^\dagger, B] + \frac{1}{2}\Sigma V \text{Tr}(\mathcal{M}^\dagger U + \mathcal{M} U^\dagger)} . \quad (2.2)$$

In (2.2) the $N_f \times N_f$ matrix

$$B = \text{diag}(\mu_1 \mathbf{1}_{N_1}, \mu_2 \mathbf{1}_{N_2}) \quad (2.3)$$

is made out of the chemical potentials, and the quark mass matrix is

$$\mathcal{M} = \text{diag}(m_1, \dots, m_{N_f}) . \quad (2.4)$$

The partition function (2.2) is a simple zero-dimensional group integral. The leading contribution to the effective low-energy field theory at finite volume V in the ϵ -regime is thus well known.

We consider now the limit in which $V \rightarrow \infty$ while $\hat{m} = m\Sigma V$ and $\hat{\mu} = \mu F_\pi \sqrt{V}$ are kept fixed. In this limit, to LO in the ϵ -expansion, the effective partition function of this theory and all the spectral correlation functions of its Dirac operator eigenvalues are completely equivalent to the chiral Random Two-Matrix Theory with imaginary chemical potential that was introduced in ref. [12]. The equivalence for the two-point function follows from [6], for all higher density correlations it was proven in [4]. Therefore, since we have proven [13] that the probability distribution of the k th smallest eigenvalue can be computed in terms of this infinite sequence of spectral correlation functions, we are free to use the chiral Random Two-Matrix Theory when performing the actual analytical computation.

As already mentioned it has been shown in [18] that also to NLO in the ϵ -expansion the random matrix expressions [12] for density correlation functions remain valid, when replacing Σ and F_π by the renormalised constants Σ_{eff} and $F_{\pi eff}$ that encode the finite-volume corrections. Only to NNLO non-universal corrections to the random matrix setting appear.

The partition function of chiral Random Two-Matrix Theory is, up to an irrelevant normalisation factor, defined as

$$\mathcal{Z}_\nu^{(N_f)} = \int d\Phi d\Psi e^{-N \text{Tr}(\Phi^\dagger \Phi + \Psi^\dagger \Psi)} \prod_{f_1=1}^{N_1} \det[\mathcal{D}_1 + m_{f_1}] \prod_{f_2=1}^{N_2} \det[\mathcal{D}_2 + m_{f_2}] \quad (2.5)$$

where $\mathcal{D}_{1,2}$ are given by

$$\mathcal{D}_{1,2} = \begin{pmatrix} 0 & i\Phi + i\mu_{1,2}\Psi \\ i\Phi^\dagger + i\mu_{1,2}\Psi^\dagger & 0 \end{pmatrix} . \quad (2.6)$$

The operator remains anti-Hermitian because the chemical potentials are imaginary, as shown explicitly. Both Φ and Ψ are complex rectangular matrices of size $N \times (N + \nu)$, where both N and ν are integers. The index ν corresponds to gauge field topology in the usual way. The aforementioned correspondence to chiral perturbation theory holds in the following microscopic large- N limit:

$$\lim_{N \rightarrow \infty} \mathcal{Z}_\nu^{(N_f)} = Z_\nu^{(N_f)} \quad \text{with} \quad \hat{m} = 2Nm, \quad \hat{\mu} = \sqrt{2N} \mu . \quad (2.7)$$

In the framework of chiral Random Two-Matrix Theory it is particularly simple to consider the situation corresponding to what we call partial quenching. Here one simply considers eigenvalues of one of the matrices, say D_1 , that then does not enter into the actual integration measure of (2.5) by setting $N_1 = 0$. In the language of the chiral Lagrangian, this needs to be done in terms of graded groups or by means of the replica method.

Referring to ref. [12] for details, we immediately write down the corresponding representation in terms of eigenvalues x_i^2 and y_i^2 of \mathcal{D}_1 and \mathcal{D}_2 , respectively,

$$\mathcal{Z}_\nu^{(N_f)} = \int_0^\infty \prod_{i=1}^N dx_i dy_i \mathcal{P}_\nu^{(N_f)}(\{x\}, \{y\}) , \quad (2.8)$$

up to an irrelevant (mass dependent) normalization factor. The integrand is the *joint probability distribution function* (jpdf), which is central for what follows:

$$\begin{aligned} \mathcal{P}_\nu^{(N_f)}(\{x\}, \{y\}) &\equiv \prod_{i=1}^N \left((x_i y_i)^{\nu+1} e^{-N(c_1 x_i^2 + c_2 y_i^2)} \prod_{f_1=1}^{N_1} (x_i^2 + m_{f_1}^2) \prod_{f_2=1}^{N_2} (y_i^2 + m_{f_2}^2) \right) \\ &\times \Delta_N(\{x^2\}) \Delta_N(\{y^2\}) \det_{1 \leq i, j \leq N} [I_\nu(2dN x_i y_j)] . \end{aligned} \quad (2.9)$$

Because the integration in eq. (2.5) was over Φ and Ψ separately, the matrices now become coupled in the exponent. The corresponding unitary group integral leads to the determinant of modified I -Bessel functions, and removes one of the initially two Vandermonde determinants, which is defined as $\Delta_N(\{x^2\}) = \prod_{j>i=1}^N (x_j^2 - x_i^2)$. The precise connection between the constants and $\mu_{1,2}$ is given by

$$\begin{aligned} c_1 &= (1 + \mu_2^2)/\delta^2 , & c_2 &= (1 + \mu_1^2)/\delta^2 , \\ d &= (1 + \mu_1 \mu_2)/\delta^2 , & \delta &= \mu_2 - \mu_1 , \\ 1 - \tau &= d^2/(c_1 c_2) , \end{aligned} \quad (2.10)$$

where the latter is defined for later convenience. We need the joint probability distribution to be normalised to unity, which is done trivially by dividing by $\mathcal{Z}_\nu^{(N_f)}$ (cf. eq. 2.8)).

3 The k th Eigenvalue at Finite- N for Arbitrary $\nu \geq 0$

We now follow the derivation of ref. [15] rather closely. We are able to do that because we focus here on the distributions of individual x -eigenvalues only - which are those we may partially quench. For that purpose it is convenient to first consider the joint probability distribution of the k smallest x -eigenvalues, ordered such that $0 \leq x_1 \leq x_2 \leq \dots \leq x_k$:

$$\Omega_\nu^{(N_f)}(x_1, \dots, x_k) \equiv \frac{N!}{\mathcal{Z}_\nu^{(N_f)}(N-k)!} \int_{x_k}^\infty dx_{k+1} \dots \int_{x_k}^\infty dx_N \int_0^\infty \prod_{i=1}^N dy_i \mathcal{P}_\nu^{(N_f)}(\{x\}, \{y\}) . \quad (3.1)$$

This quantity is then used to generate the k th x -eigenvalue distribution through the following integration¹

$$p_k^{(N_f, \nu)}(x_k) = \int_0^{x_k} dx_1 \int_{x_1}^{x_k} dx_2 \dots \int_{x_{k-2}}^{x_k} dx_{k-1} \Omega_\nu^{(N_f)}(x_1, \dots, x_k) . \quad (3.2)$$

Note that for $k=1$ no integration is needed, and $p_1^{(N_f, \nu)}(x_1) = \Omega_\nu^{(N_f)}(x_1)$.

The computation of mixed or conditional individual eigenvalue distributions, e.g. to find the joint distribution of the first x - and first y -eigenvalue, remains an open problem.

¹Compared to [15] we are already working with squared variables here. Translating to that picture the integration bounds in eqs. (3.1) and (3.2) remain the same as in [15].

We next proceed as in ref. [13], and integrate out *all* y -eigenvalues exactly. Because of this we note that in eq. (3.1) we can replace the determinant over the Bessel functions by $N!$ times its diagonal part, after having made use of the antisymmetry property of $\Delta_N(y^2)$. After inserting a representation of the Bessel function in terms of a factorised infinite sum over Laguerre polynomials (see eq. (B.7) in [12]), we get

$$\begin{aligned} \int_0^\infty \prod_{i=1}^N dy_i \mathcal{P}_\nu^{(N_f)}(\{x\}, \{y\}) &= N! \int_0^\infty \prod_{i=1}^N \left(dy_i \prod_{f_1=1}^{N_1} (x_i^2 + m_{f_1}^2) \prod_{f_2=1}^{N_2} (y_i^2 + m_{f_2}^2) \right) \Delta_N(\{x^2\}) \\ &\times \Delta_N(\{y^2\}) \prod_{i=1}^N \left((Nd)^\nu \tau^{\nu+1} (x_i y_i)^{2\nu+1} e^{-N\tau(c_1 x_i^2 + c_2 y_i^2)} \sum_{n_i=0}^\infty \frac{n_i! (1-\tau)^{n_i}}{(n_i + \nu)!} L_{n_i}^\nu(N\tau c_1 x_i^2) L_{n_i}^\nu(N\tau c_2 y_i^2) \right), \end{aligned} \quad (3.3)$$

where the Laguerre polynomials $L_j^\nu(N\tau c_2 y^2)$ now appear with their corresponding weight function $y^{2\nu+1} e^{-N\tau c_2 y^2}$ due to the identity used. Next we include the set of N_2 masses, $\{m_2\}$, into $\Delta_N(\{y^2\})$ to form a larger Vandermonde determinant of size $N + N_2$, and then replace it by a determinant of in general arbitrary Laguerre polynomials normalised to be monic

$$\Delta_N(\{y^2\}) \prod_{i=1}^N \prod_{f_2=1}^{N_2} (y_i^2 + m_{f_2}^2) = \frac{\begin{vmatrix} \hat{L}_0^{\bar{\nu}}(N\tau c_2(im_{f_2=1})^2) & \cdots & \frac{1}{(N\tau c_2)^{N+N_2-1}} \hat{L}_{N+N_2-1}^{\bar{\nu}}(N\tau c_2(im_{f_2=1})^2) \\ \vdots & \ddots & \vdots \\ \hat{L}_0^{\bar{\nu}}(N\tau c_2(im_{N_2})^2) & \cdots & \frac{1}{(N\tau c_2)^{N+N_2-1}} \hat{L}_{N+N_2-1}^{\bar{\nu}}(N\tau c_2(im_{N_2})^2) \\ \hat{L}_0^{\bar{\nu}}(N\tau c_2 y_1^2) & \cdots & \frac{1}{(N\tau c_2)^{N+N_2-1}} \hat{L}_{N+N_2-1}^{\bar{\nu}}(N\tau c_2 y_1^2) \\ \vdots & \ddots & \vdots \\ \hat{L}_0^{\bar{\nu}}(N\tau c_2 y_N^2) & \cdots & \frac{1}{(N\tau c_2)^{N+N_2-1}} \hat{L}_{N+N_2-1}^{\bar{\nu}}(N\tau c_2 y_N^2) \end{vmatrix}}{\Delta_{N_2}(\{(im_2)^2\})}. \quad (3.4)$$

Here the index of the Laguerre polynomials $\bar{\nu}$ is arbitrary. The monic Laguerre polynomials relate to ordinary Laguerre polynomials in a ν -independent manner:

$$\hat{L}_n^\nu(x) \equiv (-1)^n n! L_n^\nu(x) = \sum_{j=0}^n (-1)^{n+j} \frac{n!(n+\nu)!}{(n-j)!(\nu+j)!j!} x^j = x^n + \mathcal{O}(x^{n-1}) . \quad (3.5)$$

In eq. (3.4) the inverse powers $(N\tau c_2)^{j-1}$ can be taken out of the determinant. Inserting this back into eq. (3.3) for $\nu = \bar{\nu}$ we can use the orthogonality of the Laguerre polynomials in the integrated variables $y_{1,\dots,N}$, killing the infinite sums from the expanded Bessel functions. The Laguerre polynomials in x thus replace those in y inside the determinant, times the norm from the integration. We obtain

$$\begin{aligned} \int_0^\infty \prod_{i=1}^N dy_i \mathcal{P}_\nu^{(N_f)}(\{x\}, \{y\}) &= \\ &= \frac{N!(Nd)^{N\nu} \tau^{N(\nu+1)} \prod_{j=0}^{N+N_2-1} (1-\tau)^j (N\tau c_2)^{-j}}{\Delta_{N_2}(\{(im_2)^2\}) 2^N (N\tau c_2)^{N(\nu+1)}} \prod_{i=1}^N \left(x_i^{2\nu+1} e^{-N\tau c_1 x_i^2} \prod_{f_1=1}^{N_1} (x_i^2 + m_{f_1}^2) \right) \Delta_N(\{x^2\}) \end{aligned}$$

$$\times \begin{vmatrix} \hat{L}_0^\nu(N\tau c_2(im_{f_2=1})^2) & \cdots & \frac{1}{(1-\tau)^{N+N_2-1}} \hat{L}_{N+N_2-1}^\nu(N\tau c_2(im_{f_2=1})^2) \\ \cdots & \cdots & \cdots \\ \hat{L}_0^\nu(N\tau c_2(im_{N_2})^2) & \cdots & \frac{1}{(1-\tau)^{N+N_2-1}} \hat{L}_{N+N_2-1}^\nu(N\tau c_2(im_{N_2})^2) \\ \hat{L}_0^\nu(N\tau c_1 x_1^2) & \cdots & \hat{L}_{N+N_2-1}^\nu(N\tau c_1 x_1^2) \\ \cdots & \cdots & \cdots \\ \hat{L}_0^\nu(N\tau c_1 x_N^2) & \cdots & \hat{L}_{N+N_2-1}^\nu(N\tau c_1 x_N^2) \end{vmatrix}, \quad (3.6)$$

after taking out common factors of the determinant. The determinant in eq. (3.6), which we call D_{N+N_2} , can almost be mapped to a Vandermonde determinant, using an identity proved in appendix A in [13]

$$D_{N+N_2}(\{m_2^2\}; \{x^2\}) = \begin{vmatrix} \hat{L}_0^\nu(\frac{1}{\tau} M_{f_2=1}^2) & \cdots & \frac{\tau^{N+N_2-1}}{(1-\tau)^{N+N_2-1}} \hat{L}_{N+N_2-1}^\nu(\frac{1}{\tau} M_{f_2=1}^2) \\ \cdots & \cdots & \cdots \\ \hat{L}_0^\nu(\frac{1}{\tau} M_{N_2}^2) & \cdots & \frac{\tau^{N+N_2-1}}{(1-\tau)^{N+N_2-1}} \hat{L}_{N+N_2-1}^\nu(\frac{1}{\tau} M_{N_2}^2) \\ 1 & \cdots & X_1^{2(N+N_2-1)} \\ \cdots & \cdots & \cdots \\ 1 & \cdots & X_N^{2(N+N_2-1)} \end{vmatrix}, \quad (3.7)$$

where we have defined

$$M_{f_2}^2 \equiv N\tau c_2(im_{f_2})^2 \quad \text{and} \quad X_j^2 \equiv N\tau c_1 x_j^2. \quad (3.8)$$

This fact can be used below to perform the $N - k$ remaining integrations in the generating quantity Ω_k , after inserting eq. (3.7) into eqs. (3.6) and (3.1). This leads to

$$\begin{aligned} \Omega_\nu^{(N_f)}(x_1, \dots, x_k) &= C \prod_{j>i\geq 1}^k (x_j^2 - x_i^2) \int_{x_k}^\infty dx_{k+1} \cdots \int_{x_k}^\infty dx_N \prod_{j>i\geq k+1}^N (x_j^2 - x_i^2) \prod_{j=k+1}^N \prod_{i=1}^k (x_j^2 - x_i^2) \\ &\times \prod_{i=1}^N \left(x_i^{2\nu+1} e^{-N\tau c_1 x_i^2} \prod_{f_1=1}^{N_1} (x_i^2 + m_{f_1}^2) \right) D_{N+N_2}(\{m_2^2\}; \{x^2\}), \end{aligned} \quad (3.9)$$

where we have split the Vandermonde determinant $\Delta_N(\{x^2\})$ into integrated and unintegrated variables, and defined the following constant

$$C \equiv \frac{(N!)^2 (Nd)^{N\nu} \tau^{N(\nu+1)} \prod_{j=0}^{N+N_2-1} (1-\tau)^j (N\tau c_2)^{-j}}{\mathcal{Z}_\nu^{(N_f)}(N-k)! 2^N (N\tau c_2)^{N(\nu+1)} \Delta_{N_2}(\{(im_2)^2\})}. \quad (3.10)$$

We can now change variables $x_j \rightarrow u_j = x_j^2$ for $j = k+1, \dots, N$, and then perform the shift $u_j \rightarrow z_j = u_j - x_k^2$ to obtain integrations $\int_0^\infty dz_j$ in eq. (3.9):

$$\begin{aligned} \Omega_\nu^{(N_f)}(x_1, \dots, x_k) &= C \prod_{j>i\geq 1}^k (x_j^2 - x_i^2) \prod_{i=1}^k \left(x_i^{2\nu+1} e^{-N\tau c_1 x_i^2} \prod_{f_1=1}^{N_1} (x_i^2 + m_{f_1}^2) \right) \frac{1}{2^{(N-k)}} e^{-N(N-k)\tau c_1 x_k^2} \\ &\times \int_0^\infty \prod_{j=k+1}^N \left(dz_j z_j e^{-N\tau c_1 z_j} (z_j + x_k^2)^\nu \prod_{i=1}^{k-1} (z_j + x_k^2 - x_i^2) \prod_{f_1=1}^{N_1} (z_j + x_k^2 + m_{f_1}^2) \right) \\ &\times \prod_{j>i\geq k+1}^N (z_j - z_i) D_{N+N_2}(\{m_2^2\}; x_1^2, \dots, x_k^2, z_{k+1} + x_k^2, \dots, z_N + x_k^2). \end{aligned} \quad (3.11)$$

We thus obtain an integral with $\nu + k - 1$ extra mass terms of flavour-type “1”, in addition to the N_1 shifted masses. The weight

$$w(z) = z^1 e^{-N\tau c_1 z} \quad (3.12)$$

is now of Laguerre-type corresponding to a *fixed* topological charge of $\bar{\nu} = 1$, irrespective of the actual topological charge ν of the given gauge field sector we started with. We will therefore call $\bar{\nu}$ spurious topology. Compared with the corresponding derivation in case of vanishing chemical potential [15], this can be seen to differ by one unit, compared to spurious topology $\bar{\nu} = 2$ at vanishing $\mu_{1,2}$ in [15]. The reason for this difference is easily traced to the different integration measure for the x -eigenvalues, which has one power less in the Vandermonde determinant compared to the case of vanishing chemical potential². It is an interesting and quite non-trivial check on our present calculation that we recover the results of reference [15] in the limit of vanishing chemical potential. In particular, the shift from spurious topology $\bar{\nu} = 1$ to spurious topological charge $\bar{\nu} = 2$ in the integration measure will now arise due to recurrence relations of Laguerre polynomials. Some details of this will be given below.

When replacing the Vandermonde determinant in the variables z_j as well as D_{N+N_2} by a determinant containing Laguerre polynomials we thus choose polynomials $L_j^1(N\tau c_1 z)$ in order to be able to exploit the orthogonality properties with respect to the measure $w(z)$ eq. (3.12).

For the new masses times $\Delta_{N-k}(\{z\})$ this is an easy task. We can include them into a bigger determinant of size $N - k + N_1 + \nu + k - 1$, following the identity eq. (3.4). Here we replace the N variables y_i^2 by $N - k$ variables z_i , and the set of N_2 masses by the following set of $N_1 + \nu + k - 1$ masses:

$$\begin{aligned} m_{f_1}'^2 &\equiv m_{f_1}^2 + x_k^2 & \text{for } f_1 = 1, \dots, N_1, \\ m_{N_1+j}'^2 &\equiv x_k^2 + \epsilon_j^2 & \text{for } j = 1, \dots, \nu, \\ m_{N_1+\nu+i}'^2 &\equiv x_k^2 - x_i^2 & \text{for } i = 1, \dots, k-1, \end{aligned} \quad (3.13)$$

and likewise we define

$$M_j'^2 \equiv N\tau c_1 (im_j')^2 \quad \text{for } j = 1, \dots, N_1 + \nu + k - 1. \quad (3.14)$$

For computational simplicity we first set the ν degenerate masses to be different by adding small pairwise different constants, ϵ_j^2 , and then set $\epsilon_j = 0$ at the end of the computation. Also we may chose spurious topology $\bar{\nu} = 1$ in eq. (3.4). The prefactors in front of the Laguerre polynomials inside the determinant can be taken out.

To express the determinant D_{N+N_2} of the shifted arguments in eq. (3.11) in terms of Laguerre polynomials requires a bit more algebra:

$$\begin{aligned} D_{N+N_2}(\{m_2^2\}; x_1^2, \dots, x_k^2, z_{k+1} + x_k^2, \dots, z_N + x_k^2) = \\ = \begin{vmatrix} \hat{L}_0^\nu(\frac{1}{\tau} M_{f_2=1}^2) & \dots & \sum_{l=0}^{N+N_2-1} \frac{\tau^l}{(1-\tau)^l} \hat{L}_l^\nu(\frac{1}{\tau} M_{f_2=1}^2) (-X_k^2)^{N+N_2-1-l} \binom{N+N_2-1}{l} \\ \dots & \dots & \dots \\ \hat{L}_0^\nu(\frac{1}{\tau} M_{N_2}^2) & \dots & \sum_{l=0}^{N+N_2-1} \frac{\tau^l}{(1-\tau)^l} \hat{L}_l^\nu(\frac{1}{\tau} M_{N_2}^2) (-X_k^2)^{N+N_2-1-l} \binom{N+N_2-1}{l} \\ 1 & \dots & (X_1^2 - X_k^2)^{N+N_2-1} \\ \dots & \dots & \dots \\ 1 & \dots & (X_{k-1}^2 - X_k^2)^{N+N_2-1} \\ 1 & \dots & 0 \\ 1 & \dots & Z_{k+1}^{N+N_2-1} \\ \dots & \dots & \dots \\ 1 & \dots & Z_N^{N+N_2-1} \end{vmatrix} \end{aligned}$$

²Of course, the additional pieces due to the y -integrations are what ensures equivalence to those corresponding one-matrix model results in the limit $\mu_j \rightarrow 0$.

$$= \begin{vmatrix} q_0^\nu(M_{f_2=1}^2) & \cdots & q_{N+N_2-1}^\nu(M_{f_2=1}^2) \\ \vdots & \cdots & \vdots \\ q_0^\nu(M_{N_2}^2) & \cdots & q_{N+N_2-1}^\nu(M_{N_2}^2) \\ \hat{L}_0^1(M_{N_1+\nu+1}'^2) & \cdots & \hat{L}_{N+N_2-1}^1(M_{N_1+\nu+1}'^2) \\ \vdots & \cdots & \vdots \\ \hat{L}_0^1(M_{N_1+\nu+k}'^2) & \cdots & \hat{L}_{N+N_2-1}^1(M_{N_1+\nu+k}'^2) \\ \hat{L}_0^1(Z_{k+1}) & \cdots & \hat{L}_{N+N_2-1}^1(Z_{k+1}) \\ \vdots & \cdots & \vdots \\ \hat{L}_0^1(Z_N) & \cdots & \hat{L}_{N+N_2-1}^1(Z_N) \end{vmatrix}, \quad (3.15)$$

where for convenience we have defined $M_{N_1+\nu+k}'^2 = 0$, as well as

$$Z_k \equiv N\tau c_1 z_k. \quad (3.16)$$

In the first step in eq. (3.15) we have used the invariance of the determinant to undo the shift in x_k^2 of the z_j variables. This leads to a shift in variables X_i and to linear combinations of the Laguerre polynomials in the N_2 masses. In the second step we have added columns from the left to the right to replace monic powers in Z_j and $M_i'^2$ by polynomials \hat{L}_n^1 . Because the determinant D_{N+N_2} is not an invariant Vandermonde this leads to a further sum in the first N_2 rows, invoking the following *new polynomials*

$$q_n^\nu(M_2^2) = (-)^n n! \sum_{l=0}^n \frac{1}{(1-\tau)^l} L_l^\nu(M_2^2) L_{n-l}^{-\nu}(-X_k^2). \quad (3.17)$$

The form given here is derived in Appendix A using identities for Laguerre polynomials. For later purpose, we note already that in the limit of zero chemical potential, *i.e.* in the limit $\tau \rightarrow 0$, we obtain Laguerre polynomials of shifted mass from the q_n^ν :

$$\lim_{\tau \rightarrow 0} q_n^\nu(M_2^2) \frac{1}{(-)^n n!} = \sum_{j=0}^n L_j^\nu(-Nm_2^2) L_{n-j}^{-\nu}(-Nx_k^2) = L_n^1(-N(m_2^2 + x_k^2)). \quad (3.18)$$

In this way we recover, after the use of a few identities for Laguerre polynomials, the results of ref. [15] in the limit of vanishing chemical potential.

We now proceed with the integration over the variables z_{k+1}, \dots, z_N in eq. (3.11). Using the rewriting discussed above, we have:

$$\begin{aligned} \Omega_\nu^{(N_f)}(x_1, \dots, x_k) &= C \prod_{j>i \geq 1}^k (x_j^2 - x_i^2) \prod_{i=1}^k \left(x_i^{2\nu+1} e^{-N\tau c_1 x_i^2} \prod_{f_1=1}^{N_1} (x_i^2 + m_{f_1}^2) \right) \frac{2^{-(N-k)} e^{-N(N-k)\tau c_1 x_k^2}}{\Delta_{N_1+\nu+k-1}(\{(im')^2\})} \\ &\times \int_0^\infty \prod_{j=k+1}^N (dz_j z_j e^{-N\tau c_1 z_j}) \prod_{j=0}^{N+N_1+\nu-2} (N\tau c_1)^{-j} \\ &\times \begin{vmatrix} \hat{L}_0^1(M_1'^2) & \cdots & \hat{L}_{N+N_1+\nu-2}^1(M_1'^2) \\ \vdots & \cdots & \vdots \\ \hat{L}_0^1(M_{N_1+\nu+k-1}'^2) & \cdots & \hat{L}_{N+N_1+\nu-2}^1(M_{N_1+\nu+k-1}'^2) \\ \hat{L}_0^1(Z_{k+1}) & \cdots & \hat{L}_{N+N_1+\nu-2}^1(Z_{k+1}) \\ \vdots & \cdots & \vdots \\ \hat{L}_0^1(Z_N) & \cdots & \hat{L}_{N+N_1+\nu-2}^1(Z_N) \end{vmatrix} \begin{vmatrix} q_0^\nu(M_{f_2=1}^2) & \cdots & q_{N+N_2-1}^\nu(M_{f_2=1}^2) \\ \vdots & \cdots & \vdots \\ q_0^\nu(M_{N_2}^2) & \cdots & q_{N+N_2-1}^\nu(M_{N_2}^2) \\ \hat{L}_0^1(M_{N_1+\nu+1}'^2) & \cdots & \hat{L}_{N+N_2-1}^1(M_{N_1+\nu+1}'^2) \\ \vdots & \cdots & \vdots \\ \hat{L}_0^1(M_{N_1+\nu+k}'^2) & \cdots & \hat{L}_{N+N_2-1}^1(M_{N_1+\nu+k}'^2) \\ \hat{L}_0^1(Z_{k+1}) & \cdots & \hat{L}_{N+N_2-1}^1(Z_{k+1}) \\ \vdots & \cdots & \vdots \\ \hat{L}_0^1(Z_N) & \cdots & \hat{L}_{N+N_2-1}^1(Z_N) \end{vmatrix}. \end{aligned}$$

$$(3.19)$$

This expression can be simplified somewhat by noting the identity

$$\frac{\prod_{j>i\geq 1}^k (x_j^2 - x_i^2) \prod_{i=1}^k \left(x_i^{2\nu+1} \prod_{f_1=1}^{N_1} (x_i^2 + m_{f_1}^2) \right)}{\Delta_{N_1+\nu+k-1}(\{(im')^2\})} = \frac{x_k^{2\nu+1} \prod_{i=1}^{k-1} (x_i [x_k^2 - x_i^2]) \prod_{f_1=1}^{N_1} (x_k^2 + m_{f_1}^2)}{\Delta_{N_1}(\{(im_1)^2\}) \Delta_\nu(\{(i\epsilon)^2\}) \prod_{f_1=1}^{N_1} m_{f_1}^{2\nu}} (1 + O(\epsilon)) . \quad (3.20)$$

The Vandermonde determinant $\Delta_{N_1}(\{(im_1)^2\})$ also occurs as a factor in the partition function, and thus cancels.

The integrals over $z_j, j = k+1, \dots, N$ can be performed, again exploiting orthogonality properties of the Laguerre polynomials $L_n^1(z)$ with respect to our weight function (3.12). This is done by means of a generalisation of the original Dyson Theorem (see e.g. [25]), now for two determinants of different size, of which different sets of entries are not even of Laguerre-type, and not integrated over either. The needed generalisation of this theorem was essentially provided in ref. [26], and we only need a slight extension of this more general theorem here. We have relegated the proof of the new theorem to Appendix B, and will only quote the result here.

To express the result, we define two kernels (both corresponding to the kernel K^I of Appendix B for different values of the indices)

$$K(M_1'^2, M_2'^2) \equiv \sum_{j=0}^{\Lambda} \frac{1}{j+1} L_j^1(M_1'^2) L_j^1(M_2'^2) , \quad \tilde{K}^\nu(M'^2, M^2) \equiv \sum_{j=0}^{\Lambda} \frac{(-)^j}{(j+1)!} L_j^1(M'^2) q_j^\nu(M^2) , \quad (3.21)$$

with $\Lambda = \min(N + N_1 + \nu - 2, N + N_2 - 1)$, and the associated matrix of size $(N_2 + k) \times (N_1 + \nu + k - 1)$

$$\mathcal{K} \equiv \begin{pmatrix} \tilde{K}^\nu(M_1'^2, M_{f_2=1}^2) & \cdots & \tilde{K}^\nu(M_{N_1+\nu+k-1}'^2, M_{f_2=1}^2) \\ \cdots & \cdots & \cdots \\ \tilde{K}^\nu(M_1'^2, M_{N_2}^2) & \cdots & \tilde{K}^\nu(M_{N_1+\nu+k-1}'^2, M_{N_2}^2) \\ K(M_1'^2, M_{N_1+\nu+1}'^2) & \cdots & K(M_{N_1+\nu+k-1}'^2, M_{N_1+\nu+1}'^2) \\ \cdots & \cdots & \cdots \\ K(M_1'^2, M_{N_1+\nu+k}'^2) & \cdots & K(M_{N_1+\nu+k-1}'^2, M_{N_1+\nu+k}'^2) \end{pmatrix} . \quad (3.22)$$

We can now give the answer for $\Omega_\nu^{(N_f)}$, where we have to distinguish 3 different cases, depending on the respective sizes of the two determinants in (3.19).

- i) The simplest case is $N_2 = N_1 + \nu - 1$ (this implies that \mathcal{K} is square). Applying Appendix B to the integral of (3.19) and using eq. (3.20) we obtain to leading order in mass difference ϵ_j :

$$\begin{aligned} \Omega_\nu^{(N_f)}(x_1, \dots, x_k) &= C \frac{x_k^{2\nu+1} \prod_{i=1}^{k-1} (x_i e^{-N\tau c_1 x_i^2} [x_k^2 - x_i^2]) \prod_{f_1=1}^{N_1} (x_k^2 + m_{f_1}^2)}{2^{N-k} \Delta_{N_1}(\{(im_1)^2\}) \Delta_\nu(\{(i\epsilon)^2\}) \prod_{f_1=1}^{N_1} m_{f_1}^{2\nu}} (N\tau c_1)^{-2(N-k)} \\ &\times e^{-N(N-k+1)\tau c_1 x_k^2} \prod_{j=0}^{N+N_1+\nu-2} (N\tau c_1)^{-j} \prod_{j=0}^{\Lambda} \left((j+1)! j! \right) (N-k)! (-)^{(N_1+N_2+\nu+1)(N-k)} \det \mathcal{K} \end{aligned} \quad (3.23)$$

A typical example for this case would be $N_1 = N_2 = \nu = 1$.

- ii) For $N_2 > N_1 + \nu - 1$ the expression is the same, but with additional columns in the determinant in the second line of eq. (3.23):

$$\left| \begin{array}{cccc} \mathcal{K} & q_{N+N_1+\nu-1}^\nu(M_{f_2}^2) & \cdots & q_{N+N_2-1}^\nu(M_{f_2}^2) \\ \hat{L}_{N+N_1+\nu-1}^1(M_{N_1+\nu+j}'^2) & \cdots & \hat{L}_{N+N_2-1}^1(M_{N_1+\nu+j}'^2) & \end{array} \right|, \quad f_2 = 1, \dots, N_2, \quad j = 1, \dots, k. \quad (3.24)$$

This case typically occurs for partial quenching $N_1 = 0$, $N_2 = 2$ and small topology $\nu = 0, 1, 2$

- iii) Similarly, for $N_2 < N_1 + \nu - 1$, which typically occurs for higher topology, the determinant in eq. (3.23) is replaced by

$$\left| \begin{array}{cccc} \mathcal{K}^T & \hat{L}_{N+N_2}^1(M_j'^2) & \cdots & \hat{L}_{N+N_1+\nu-2}^1(M_j'^2) \end{array} \right|, \quad j = 1, \dots, N_1 + \nu + k - 1. \quad (3.25)$$

We note that all three cases contain degeneracies for $\nu > 1$, and that a Taylor expansion must be performed when sending $\epsilon_j \rightarrow 0$.

Finally we are ready to compute the k th individual eigenvalue probability distribution $p_k^{(\nu)}(x)$ by means of inserting the above results into eq. (3.2)

$$p_k^{(N_f, \nu)}(x_k) = \int_0^{x_k} dx_1 \int_{x_1}^{x_k} dx_2 \cdots \int_{x_{k-2}}^{x_k} dx_{k-1} \Omega_\nu^{(N_f)}(x_1, \dots, x_k).$$

An alternative, equivalent formulation for the first eigenvalue $p_{k=1}^{(\nu)}(x)$ at arbitrary ν follows from the derivative of its cumulative distribution given in appendix C, where we generalise the approach of [13] to $\nu > 0$.

We refrain from giving further explicit examples for finite N and instead turn to the large- N limit.

4 The Microscopic Scaling Limit

Having obtained the explicit solution for any finite N , we are now ready to take the appropriate microscopic $N \rightarrow \infty$ scaling limit. In the language of QCD this corresponds to a finite-volume scaling in V , where V is the space-time volume. It is only in this limit that we expect to obtain universal results which do not depend on having chosen Gaussian measures for the two original matrices Φ and Ψ . We remind the reader that the appearance of Laguerre polynomials in the finite- N solution is directly linked to these two original measures having been chosen Gaussian. As we have seen, there are numerous instances where our derivation makes explicit reference to specific identities for Laguerre polynomials, and also the measure factor was specific to these Gaussian integrals. It is therefore a quite non-trivial task, and an interesting challenge, to generalise the universality proof of the last reference in [2] to this more general setting. The universality has been implicitly checked by the equivalence proof of [4], but an explicit proof of universality directly in the framework of the chiral Random Two-Matrix Theory remains to be constructed.

In QCD-terminology, we keep $\hat{m}_i \equiv m_i \Sigma V$ and $\hat{\mu} \equiv \mu F_\pi \sqrt{V}$ fixed, while we take $V \rightarrow \infty$. In the language of our chiral Random Two-Matrix Theory, we take the $N \rightarrow \infty$ limit while keeping

$$\hat{x}_i \equiv 2N x_i, \quad \hat{m}_i \equiv 2N m_i, \quad \hat{\mu}_{1,2}^2 \equiv 2N \mu_{1,2}^2 \quad (4.1)$$

fixed (we also scale the masses ϵ_j in the same way, before taking them to zero at the end of the calculation). In addition, we introduce the following relevant quantity, the *difference* in rescaled

chemical potential, $\hat{\delta} \equiv \hat{\mu}_2 - \hat{\mu}_1$. We will follow [13] closely, without giving a detailed derivation. Defining the following continuum indices

$$t \equiv j/N, \text{ and } r \equiv l/j, \quad (4.2)$$

we replace the sum by an integral, $\sum_{j=0}^{\Lambda} N^{-1} \rightarrow \int_0^1 dt$, and correspondingly for index l . In addition, we use the identities

$$\lim_{N,j,l \rightarrow \infty} \frac{1}{(1-\tau)^l} = \exp \left[\frac{1}{2} r t \hat{\delta}^2 \right], \quad \lim_{N,j,l \rightarrow \infty} j^{-\nu} L_l^{\nu}(M'^2 = -N\tau c_1 m^2) = \left(\frac{4r}{\hat{m}^2 t} \right)^{\nu/2} I_{\nu}(\sqrt{rt}\hat{m}). \quad (4.3)$$

The scaling limit of the kernel K is obtained using the Christoffel-Darboux identity:

$$\begin{aligned} K_S(\hat{m}'_1, \hat{m}'_2) &\equiv \lim_{N \rightarrow \infty} \frac{1}{N^2} K_N(M_1'^2, M_2'^2) \\ &= \lim_{N \rightarrow \infty} -\frac{1}{N^2} \frac{L_{N+1}^1(M_1'^2) L_N^1(M_2'^2) - L_{N+1}^1(M_2'^2) L_N^1(M_1'^2)}{M_1'^2 - M_2'^2} \\ &= \frac{8}{\hat{m}'_1 \hat{m}'_2} \frac{\hat{m}'_1 I_0(\hat{m}'_1) I_1(\hat{m}'_2) - \hat{m}'_2 I_0(\hat{m}'_2) I_1(\hat{m}'_1)}{\hat{m}'_1{}^2 - \hat{m}'_2{}^2} \end{aligned} \quad (4.4)$$

The scaling limit of the new polynomials q_N^{ν} requires a little more care. For $\nu > 0$ one needs to treat the terms corresponding to indices $l = N - \nu + 1, \dots, N$ in (3.17) separately, due to the presence of Laguerre polynomials with negative index³ [13]. For these “anomalous” terms, the scaling limit is

$$\begin{aligned} q_{SA}^{\nu}(\hat{m}; t) &\equiv \lim_{N,j \rightarrow \infty} j^{-1} \sum_{l=j-\nu+1}^j \frac{1}{(1-\tau)^l} L_l^{\nu}(M^2) L_{j-l}^{-\nu}(-X_k^2) \\ &= \lim_{N,j \rightarrow \infty} \sum_{p=0}^{\nu-1} \frac{j^{\nu-p-1}}{p!} \left(-\frac{t\hat{x}_k^2}{4} + O(j^{-1}) \right)^p \sum_{l=j-\nu+1}^{j-p} (-)^{j-l} \binom{\nu-p-1}{j-l-p} \frac{1}{(1-\tau)^l} j^{-\nu} L_l^{\nu}(M^2) \\ &= \sum_{p=0}^{\nu-1} \frac{1}{p!} \left(\frac{t\hat{x}_k^2}{4} \right)^p \left. \frac{\partial^{\nu-p-1}}{\partial r^{\nu-p-1}} \right|_{r=1} \left[e^{\frac{1}{2} r t \hat{\delta}^2} \left(\frac{4r}{t\hat{m}^2} \right)^{\nu/2} I_{\nu}(\sqrt{rt}\hat{m}) \right]. \end{aligned} \quad (4.5)$$

Here we write out $L_{j-l}^{-\nu}$, and use the fact that the binomial weight kills the terms of order lower than $\nu - p - 1$ of the expansion in r . Note that naively q_{SA}^{ν} is proportional to $j^{\nu-1}$, but that the pattern of cancelation is exactly such that the limit is finite. The notation in eq. (4.5) is chosen such that for $\nu = 0$ the sum is void, and thus the anomalous term is absent for $\nu = 0$, $q_{SA}^{\nu=0} = 0$.

For the remainder of the terms, the sum turns into an integral:

$$\begin{aligned} q_{SR}^{\nu}(\hat{m}; t) &\equiv \lim_{N,j \rightarrow \infty} j^{-1} \sum_{l=0}^{j-\nu} \frac{1}{(1-\tau)^l} L_l^{\nu}(M^2) L_{j-l}^{-\nu}(-X_k^2) \\ &= \left(\frac{\hat{x}}{\hat{m}} \right)^{\nu} \int_0^1 dr e^{\frac{1}{2} r t \hat{\delta}^2} \left(\frac{r}{1-r} \right)^{\nu/2} I_{\nu}(\sqrt{rt}\hat{m}) I_{\nu}(\sqrt{(1-r)t}\hat{x}). \end{aligned} \quad (4.6)$$

The final scaling limit of the new polynomials is then the sum

$$q_S^{\nu}(\hat{m}; t) \equiv \lim_{N,j \rightarrow \infty} \frac{(-)^j}{j! j} q_j^{\nu}(M^2) = q_{SR}^{\nu}(\hat{m}; t) + q_{SA}^{\nu}(\hat{m}; t). \quad (4.7)$$

³For integer $\nu > 0$ and fixed j, z , the limit $\lim_{n \rightarrow \infty} n^{\nu} L_j^{-\nu}(z/n)$ is only finite for $j \geq \nu$.

Let us give the limiting polynomials for the first few topologies $\nu = 0, 1, 2$ as examples:

$$q_S^{\nu=0}(\hat{m}; t) = q_{SR}^{\nu=0}(\hat{m}; t) = \int_0^1 dr e^{\frac{1}{2}rt\hat{\delta}^2} I_0(\sqrt{rt}\hat{m}) I_0(\sqrt{(1-r)t}\hat{x}), \quad (4.8)$$

$$q_S^{\nu=1}(\hat{m}; t) = \frac{\hat{x}}{\hat{m}} \int_0^1 dr e^{\frac{1}{2}rt\hat{\delta}^2} \left(\frac{r}{1-r} \right)^{1/2} I_1(\sqrt{rt}\hat{m}) I_1(\sqrt{(1-r)t}\hat{x}) + e^{\frac{1}{2}t\hat{\delta}^2} \frac{2}{\sqrt{t}\hat{m}} I_1(\sqrt{t}\hat{m}), \quad (4.9)$$

$$q_S^{\nu=2}(\hat{m}; t) = \left(\frac{\hat{x}}{\hat{m}} \right)^2 \int_0^1 dr e^{\frac{1}{2}rt\hat{\delta}^2} \frac{r}{1-r} I_2(\sqrt{rt}\hat{m}) I_2(\sqrt{(1-r)t}\hat{x}) + e^{\frac{1}{2}t\hat{\delta}^2} \left(\frac{2\hat{\delta}^2 + \hat{x}^2}{\hat{m}^2} I_2(\sqrt{t}\hat{m}) + \frac{2}{\sqrt{t}\hat{m}} I_1(\sqrt{t}\hat{m}) \right). \quad (4.10)$$

For comparison see eq. (C.12) for the limiting polynomials of the previous approach [13] for the first eigenvalue extended to $\nu \geq 0$. Because of technical reasons the anomalous terms appear there already at $\nu = 0$.

The last building block we need is the new kernel $\tilde{K}^\nu(M'^2, M^2)$, eq. (3.21). It contains both the regular and anomalous terms $q_S^\nu = q_{SR}^\nu + q_{SA}^\nu$ inside the integral (originating from the sum), but no further complications occur. Hence we have the scaling limit

$$\tilde{K}_S^\nu(\hat{m}', \hat{m}) \equiv \lim_{N \rightarrow \infty} \frac{1}{N^2} \tilde{K}_N^\nu(M'^2, M^2) = \frac{2}{\hat{m}'} \int_0^1 dt \sqrt{t} I_1(\sqrt{t}\hat{m}') q_S^\nu(\hat{m}; t). \quad (4.11)$$

Using these building blocks we can calculate the main quantity of interest, the scaling limit of the eigenvalue distributions, defined as

$$p_{S_k}^{(N_f, \nu)}(\hat{x}_k) \equiv \int_0^{\hat{x}_k} d\hat{x}_1 \int_{\hat{x}_1}^{\hat{x}_k} d\hat{x}_2 \dots \int_{\hat{x}_{k-2}}^{\hat{x}_k} d\hat{x}_{k-1} \Omega_{S_\nu}^{(N_f)}(\hat{x}_1, \dots, \hat{x}_k), \quad (4.12)$$

with

$$\Omega_{S_\nu}^{(N_f)}(\hat{x}_1, \dots, \hat{x}_k) \equiv \lim_{N \rightarrow \infty} (2N)^{-k} \Omega_\nu^{(N_f)} \left(x_1 = \frac{\hat{x}_1}{2N}, \dots, x_k = \frac{\hat{x}_k}{2N} \right). \quad (4.13)$$

4.1 Example partial quenching

As pointed out in [5], there is no $\hat{\delta}$ dependence in the fully quenched ($N_1 = N_2 = 0$) theory. The simplest case to consider is then also what is probably the physically most relevant situation, namely the partially quenched case, where $N_1 = 0$. On the lattice gauge theory side, this corresponds to generating the gauge configurations using N_2 dynamical quarks with zero chemical potential, and then looking at the spectrum of the Dirac operator \mathcal{D}_1 with $\hat{\mu}_1 = -\hat{\delta}$.

In the simplest case $N_2 = 1$, $\nu = 0$ and $k = 1$ (case *ii*), the matrix \mathcal{K} is empty, and (3.24) is just a 2×2 determinant of polynomials, with $M'_1 = 0$. The two columns are degenerate, so we perform a Taylor-expansion in t . With the abbreviation $\hat{m} = \hat{m}_{f_2=1}$, the limiting distribution is then found to be

$$p_{S_1}^{(0+1,0)}(\hat{x}_1) = -\frac{e^{-\frac{1}{2}\hat{\delta}^2}}{2I_0(\hat{m})} \hat{x}_1 e^{-\frac{1}{4}\hat{x}_1^2} \begin{vmatrix} q_S^0(\hat{m}; t=1) & \partial_t q_S^0(\hat{m}; t)|_{t=1} \\ 1 & 0 \end{vmatrix} \quad (4.14)$$

$$= \frac{\hat{x}_1 e^{-\frac{1}{2}\hat{\delta}^2 - \frac{1}{4}\hat{x}_1^2}}{2I_0(\hat{m})} \partial_t q_S^0(\hat{m}; t)|_{t=1}, \quad (4.15)$$

in agreement (after partial integration) with the expression in [5]. By introducing primed masses, and changing the index of the (scaling limit of the) new polynomials q_S and the partition function, we obtain the higher topology distributions which are new.

For $\nu = 1$ (case *ii*), we can set $\epsilon_1 = 0$ in eq. (3.13) from the beginning, and we find

$$p_{S1}^{(0+1,1)}(\hat{x}_1) = -\frac{\hat{m}\hat{x}_1^3 e^{-\frac{1}{4}\hat{x}_1^2 - \frac{1}{2}\hat{\delta}^2}}{16I_1(\hat{m})} \begin{vmatrix} \tilde{K}_S^1(\hat{x}_1, \hat{m}) & q_S^1(\hat{m}; t=1) \\ K_S(\hat{x}_1, 0) & 1 \end{vmatrix}, \quad (4.16)$$

where we note that the singularities of $K(\hat{m}'_1, \hat{m}'_2)$ at $\hat{m}'_2 \rightarrow \hat{m}'_1$ and $\hat{m}'_2 \rightarrow 0$ are removable. Increasing the topology further the primed masses become degenerate, so we take the scaling limit with the (scaled) ϵ_j 's to be finite, and then let $\epsilon_j \rightarrow 0$. The Vandermonde $\Delta_\nu(\{(i\epsilon)^2\})$ of (3.23) ensures that this limit is nontrivial. We then find for $\nu = 2$ (case *i*),

$$p_{S1}^{(0+1,2)}(\hat{x}_1) = -\frac{\hat{m}^2 \hat{x}_1^4 e^{-\frac{1}{4}\hat{x}_1^2 - \frac{1}{2}\hat{\delta}^2}}{64I_2(\hat{m})} \begin{vmatrix} \tilde{K}_S^2(\hat{x}_1, \hat{m}) & \partial_{\hat{m}'} \tilde{K}_S^2(\hat{x}_1, \hat{m}) \\ K_S(\hat{x}_1, 0) & \partial_{\hat{m}'} K_S(\hat{x}_1, 0) \end{vmatrix}, \quad (4.17)$$

and for $\nu = 3$ (case *iii*),

$$p_{S1}^{(0+1,3)}(\hat{x}_1) = -\frac{\hat{m}^3 \hat{x}_1^4 e^{-\frac{1}{4}\hat{x}_1^2 - \frac{1}{2}\hat{\delta}^2}}{64I_3(\hat{m})} \begin{vmatrix} \tilde{K}_S^3(\hat{x}_1, \hat{m}) & \partial_{\hat{m}'} \tilde{K}_S^3(\hat{x}_1, \hat{m}) & \partial_{\hat{m}'}^2 \tilde{K}_S^3(\hat{x}_1, \hat{m}) \\ K_S(\hat{x}_1, 0) & \partial_{\hat{m}'} K_S(\hat{x}_1, 0) & \partial_{\hat{m}'}^2 K_S(\hat{x}_1, 0) \\ \frac{1}{\hat{x}_1} I_1(\hat{x}_1) & \frac{1}{\hat{x}_1} I_2(\hat{x}_1) & \partial_{\hat{x}_1} [\frac{1}{\hat{x}_1} I_2(\hat{x}_1)] \end{vmatrix}, \quad (4.18)$$

where by $\partial_{\hat{m}'}$ we mean the derivative with respect to the first argument (which, incidentally, is not the same as $\partial_{\hat{x}_1}$). Our new results for $\nu = 1, 2$ and 3 are illustrated in figure 1. For alternative expressions in an equivalent formulation see appendix C. The first eigenvalue distribution alone is in general not sufficient to fit both low-energy constants (LEC) Σ and F_π , see the discussion in [22]. The benefit of computing the first eigenvalue distribution at higher topology here is to be able to fit both LEC independently from different lattice configurations, where typically $\nu = 1$ offers better statistics than $\nu = 0$.

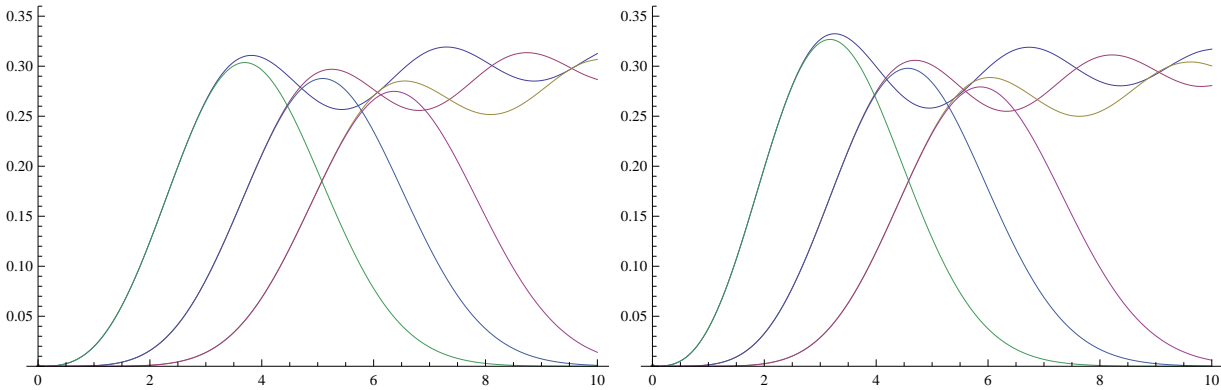


Figure 1: The first eigenvalue in the partially quenched case with $N_1 = 0$ and $N_2 = 1$ at mass $\hat{m} = 3$, shown for 3 different topologies and 2 different values of $\hat{\delta} = 1$ (left figure) and $\hat{\delta} = 3$ (right figure). The left, middle and right curves correspond to topology $\nu = 1, 2$ and 3 , respectively. The effect of the increasing number of exact zero-eigenvalues can be seen nicely. We also show the corresponding spectral density from [12] for comparison (see eq. (C.18) in appendix C). The first eigenvalue always follows the density almost up to its first maximum for all parameter values, which is an important consistency check.

Continuing with the new expressions for the k th lowest eigenvalues for $N_f = 0 + 1$ with $k > 1$, we will, for simplicity, focus on $\nu = 0$. Results at higher topology are easily obtained following the same approach as above. For $k = 2$ and $k = 3$ the joint probability distribution is found to be respectively

$$\Omega_{S0}^{(0+1)}(\hat{x}_1, \hat{x}_2) = -\frac{e^{-\frac{1}{2}\hat{\delta}^2}}{16I_0(\hat{m})}\hat{x}_1\hat{x}_2(\hat{x}_2^2 - \hat{x}_1^2)e^{-\frac{1}{4}\hat{x}_2^2} \begin{vmatrix} \tilde{K}_S^0(\hat{x}_{21}, \hat{m}) & q_S^0(\hat{m}; t=1) & \partial_t q_S^0(\hat{m}; t)|_{t=1} \\ K_S(\hat{x}_{21}, \hat{x}_{21}) & \frac{2}{\hat{x}_{21}}I_1(\hat{x}_{21}) & I_2(\hat{x}_{21}) \\ K_S(\hat{x}_{21}, 0) & 1 & 0 \end{vmatrix}. \quad (4.19)$$

and

$$\Omega_{S0}^{(0+1)}(\hat{x}_1, \hat{x}_2, \hat{x}_3) = -\frac{e^{-\frac{1}{2}\hat{\delta}^2}}{128I_0(\hat{m})}\hat{x}_1\hat{x}_2\hat{x}_3(\hat{x}_3^2 - \hat{x}_1^2)(\hat{x}_3^2 - \hat{x}_2^2)e^{-\frac{1}{4}\hat{x}_3^2} \times \begin{vmatrix} \tilde{K}_S^0(\hat{x}_{31}, \hat{m}) & \tilde{K}_S^0(\hat{x}_{32}) & q_S^0(\hat{m}; t=1) & \partial_t q_S^0(\hat{m}; t)|_{t=1} \\ K_S(\hat{x}_{31}, \hat{x}_{31}) & K_S(\hat{x}_{32}, \hat{x}_{31}) & \frac{2}{\hat{x}_{31}}I_1(\hat{x}_{31}) & I_2(\hat{x}_{31}) \\ K_S(\hat{x}_{31}, \hat{x}_{32}) & K_S(\hat{x}_{32}, \hat{x}_{32}) & \frac{2}{\hat{x}_{32}}I_1(\hat{x}_{32}) & I_2(\hat{x}_{32}) \\ K_S(\hat{x}_{31}, 0) & K_S(\hat{x}_{32}, 0) & 1 & 0 \end{vmatrix}. \quad (4.20)$$

Here we have used the shorthand $\hat{x}_{ij} \equiv \sqrt{\hat{x}_i^2 - \hat{x}_j^2}$. The eigenvalue distributions then follow by applying eq. (4.12).

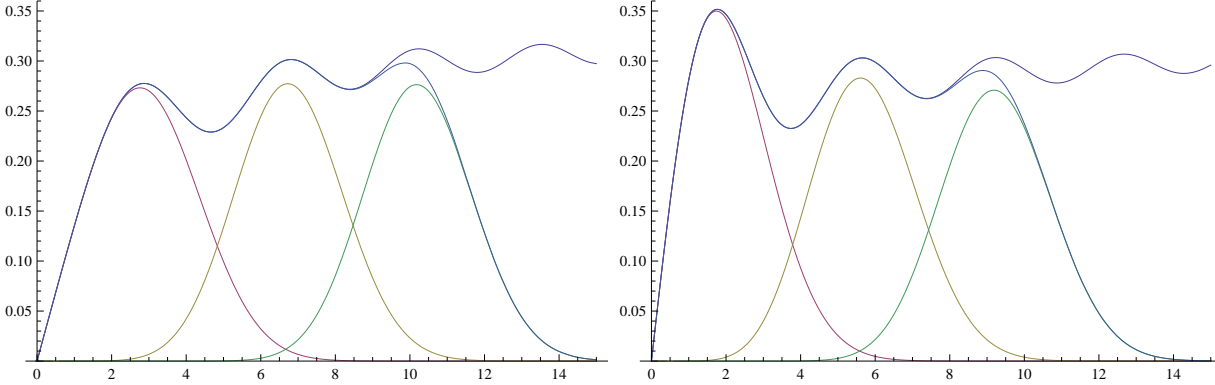


Figure 2: The first, second and third eigenvalue $k = 1, 2, 3$ in the partially quenched case with $N_1 = 0$ and $N_2 = 2$ at masses $\hat{m}_1 = 3$ and $\hat{m}_1 = 4$, all at topology $\nu = 0$. We display 2 different values of $\hat{\delta} = 1$ (left figure) and $\hat{\delta} = 3$ (right figure) to illustrate the influence of chemical potential. The left, middle and right curves correspond to $k = 1, 2$ and 3 , respectively. Also shown is the corresponding spectral density from [12] (see eq. (C.24) in appendix C). It is compared to the sum of the first three eigenvalue distributions, following the density identically up to the third eigenvalue for all parameter values.

The extension of these results to higher values of N_2 is straightforward. We will restrict ourselves here to $N_2 = 2$ and $\nu = 0$. Defining the normalisation constant

$$\mathcal{N}^{(0+2)} = e^{\hat{\delta}^2} \begin{vmatrix} I_0(\hat{m}_1) & \hat{m}_1 I_1(\hat{m}_1) \\ I_0(\hat{m}_2) & \hat{m}_2 I_1(\hat{m}_2) \end{vmatrix}, \quad (4.21)$$

we have

$$p_{S1}^{(0+2,0)}(\hat{x}_1) = \frac{\hat{x}_1 e^{-\frac{1}{4}\hat{x}_1^2}}{\mathcal{N}^{(0+2)}} \begin{vmatrix} \partial_t q_S^0(\hat{m}_1; t)|_{t=1} & \partial_t^2 q_S^0(\hat{m}_1; t)|_{t=1} \\ \partial_t q_S^0(\hat{m}_2; t)|_{t=1} & \partial_t^2 q_S^0(\hat{m}_2; t)|_{t=1} \end{vmatrix}. \quad (4.22)$$

Note that an alternative representation for the integral of this quantity, its cumulative distribution, was given in eq. (4.44) in [13].

For $k = 2$ and $k = 3$ we obtain the following new expressions

$$\Omega_{S0}^{(0+2)}(\hat{x}_1, \hat{x}_2) = -\frac{\hat{x}_1 \hat{x}_2 (\hat{x}_2^2 - \hat{x}_1^2) e^{-\frac{1}{4}\hat{x}_2^2}}{8\mathcal{N}^{(0+2)}} \times \begin{vmatrix} \tilde{K}_S^0(\hat{x}_{21}, \hat{m}_1) & q_S^0(\hat{m}_1; t=1) & \partial_t q_S^0(\hat{m}_1; t)|_{t=1} & \partial_t^2 q_S^0(\hat{m}_1; t)|_{t=1} \\ \tilde{K}_S^0(\hat{x}_{21}, \hat{m}_2) & q_S^0(\hat{m}_2; t=1) & \partial_t q_S^0(\hat{m}_2; t)|_{t=1} & \partial_t^2 q_S^0(\hat{m}_2; t)|_{t=1} \\ K_S(\hat{x}_{21}, \hat{x}_{21}) & \frac{2}{\hat{x}_{21}} I_1(\hat{x}_{21}) & I_2(\hat{x}_{21}) & \frac{\hat{x}_{21}}{2} I_3(\hat{x}_{21}) \\ K_S(\hat{x}_{21}, 0) & 1 & 0 & 0 \end{vmatrix}, \quad (4.23)$$

and

$$\Omega_{S0}^{(0+2)}(\hat{x}_1, \hat{x}_2, \hat{x}_3) = \frac{\hat{x}_1 \hat{x}_2 \hat{x}_3 (\hat{x}_3^2 - \hat{x}_1^2)(\hat{x}_3^2 - \hat{x}_2^2) e^{-\frac{1}{4}\hat{x}_3^2}}{2^6 \mathcal{N}^{(0+2)}} \times \begin{vmatrix} \tilde{K}_S^0(\hat{x}_{31}, \hat{m}_1) & \tilde{K}_S^0(\hat{x}_{32}, \hat{m}_1) & q_S^0(\hat{m}_1; t=1) & \partial_t q_S^0(\hat{m}_1; t)|_{t=1} & \partial_t^2 q_S^0(\hat{m}_1; t)|_{t=1} \\ \tilde{K}_S^0(\hat{x}_{31}, \hat{m}_2) & \tilde{K}_S^0(\hat{x}_{32}, \hat{m}_2) & q_S^0(\hat{m}_2; t=1) & \partial_t q_S^0(\hat{m}_2; t)|_{t=1} & \partial_t^2 q_S^0(\hat{m}_2; t)|_{t=1} \\ K_S(\hat{x}_{31}, \hat{x}_{31}) & K_S(\hat{x}_{32}, \hat{x}_{31}) & \frac{2}{\hat{x}_{31}} I_1(\hat{x}_{31}) & I_2(\hat{x}_{31}) & \frac{\hat{x}_{31}}{2} I_3(\hat{x}_{31}) \\ K_S(\hat{x}_{31}, \hat{x}_{32}) & K_S(\hat{x}_{32}, \hat{x}_{32}) & \frac{2}{\hat{x}_{32}} I_1(\hat{x}_{32}) & I_2(\hat{x}_{32}) & \frac{\hat{x}_{32}}{2} I_3(\hat{x}_{32}) \\ K_S(\hat{x}_{31}, 0) & K_S(\hat{x}_{32}, 0) & 1 & 0 & 0 \end{vmatrix}, \quad (4.24)$$

after inserting them into eq. (4.12). We illustrate the distributions of individual eigenvalues following from these equations in figure 2 for non-degenerate masses. In the case of equal masses, two rows become degenerate in both the numerator and denominator, necessitating a Taylor expansion. Figure 2 offers a further graphical consistency check for our results, when comparing the sum of the individual eigenvalues to the actual density. The two curves nicely agree almost up to the third local maximum for all parameter values.

4.2 Two light flavours

For $N_1 = N_2 = 1$, $\nu = 0$ the distribution of the first eigenvalue is given by

$$p_{S1}^{(1+1,0)}(\hat{x}_1) = -\frac{\hat{x}_1(\hat{m}_1^2 + \hat{x}_1^2) e^{-\frac{1}{4}\hat{x}_1^2}}{8\mathcal{N}^{(1+1)}} \begin{vmatrix} \tilde{K}_S^0(\hat{m}', \hat{m}_2) & q_S^0(\hat{m}_2; t=1) \\ K_S(\hat{m}', 0) & 1 \end{vmatrix}, \quad (4.25)$$

$$\mathcal{N}^{(1+1)} = \int_0^1 dt e^{\frac{1}{2}t\delta^2} I_0(\sqrt{t}\hat{m}_1) I_0(\sqrt{t}\hat{m}_2), \quad (4.26)$$

with $\hat{m}_1 = \hat{m}_{f_1=1}$, $\hat{m}_2 = \hat{m}_{f_1=2}$ and $\hat{m}' = \sqrt{\hat{m}_1^2 + \hat{x}_k^2}$. For its cumulative distribution see eq. (4.33) in [13].

The expressions for distribution of the second and third eigenvalue which are new results follow in a similar fashion from

$$\Omega_{S0}^{(1+1)}(\hat{x}_1, \hat{x}_2) = \frac{\hat{x}_1 \hat{x}_2 (\hat{x}_2^2 - \hat{x}_1^2)(\hat{m}_1^2 + \hat{x}_2^2) e^{-\frac{1}{4}\hat{x}_2^2}}{64\mathcal{N}^{(1+1)}} \begin{vmatrix} \tilde{K}_S^0(\hat{m}', \hat{m}_2) & \tilde{K}_S^0(\hat{x}_{21}, \hat{m}_2) & q_S^0(\hat{m}_2; t=1) \\ K_S(\hat{m}', \hat{x}_{21}) & K_S(\hat{x}_{21}, \hat{x}_{21}) & \frac{2}{\hat{x}_{21}} I_1(\hat{x}_{21}) \\ K_S(\hat{m}', 0) & K_S(\hat{x}_{21}, 0) & 1 \end{vmatrix}, \quad (4.27)$$

and

$$\Omega_{S_0}^{(1+1)}(\hat{x}_1, \hat{x}_2, \hat{x}_3) = -\frac{\hat{x}_1 \hat{x}_2 \hat{x}_3 (\hat{x}_3^2 - \hat{x}_1^2)(\hat{x}_3^2 - \hat{x}_2^2)(\hat{m}_1^2 + \hat{x}_3^2)e^{-\frac{1}{4}\hat{x}_3^2}}{2^9 \mathcal{N}^{(1+1)}} \times \begin{vmatrix} \tilde{K}_S^0(\hat{m}', \hat{m}_2) & \tilde{K}_S^0(\hat{x}_{31}, \hat{m}_2) & \tilde{K}_S^0(\hat{x}_{32}, \hat{m}_2) & q_S^0(\hat{m}_2; t=1) \\ K_S(\hat{m}', \hat{x}_{31}) & K_S(\hat{x}_{31}, \hat{x}_{31}) & K_S(\hat{x}_{32}, \hat{x}_{31}) & \frac{2}{\hat{x}_{31}} I_1(\hat{x}_{31}) \\ K_S(\hat{m}', \hat{x}_{32}) & K_S(\hat{x}_{31}, \hat{x}_{32}) & K_S(\hat{x}_{32}, \hat{x}_{32}) & \frac{2}{\hat{x}_{32}} I_1(\hat{x}_{32}) \\ K_S(\hat{m}', 0) & K_S(\hat{x}_{31}, 0) & K_S(\hat{x}_{32}, 0) & 1 \end{vmatrix}. \quad (4.28)$$

The corresponding figures are shown in figure 3.

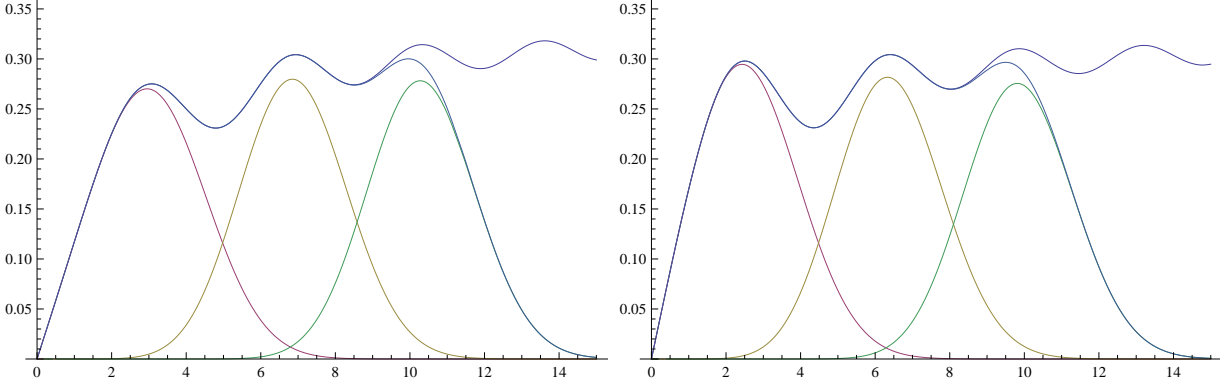


Figure 3: The first, second and third eigenvalue $k = 1, 2, 3$ for two light flavours with $N_1 = 1$ at mass $\hat{m}_1 = 3$ and $N_2 = 1$ at mass $\hat{m}_1 = 4$, all at topology $\nu = 0$. We display 2 different values of $\hat{\delta} = 1$ (left figure) and $\hat{\delta} = 3$ (right figure) to illustrate the influence of chemical potential. The left, middle and right curves correspond to $k = 1, 2$ and 3 , respectively. The corresponding spectral density is taken from [12] (see eq. (C.22) in appendix C). Again we find an excellent agreement between the density and the sum of the three individual eigenvalues for all parameter values chosen.

Both figures 2 and 3 illustrate the influence of chemical potential and thus the possibility to determine F_π from individual eigenvalue distributions. In particular the shape of the first eigenvalue, but also the positions of the maxima of the second and third eigenvalue change considerably when comparing the left and right plots, in particular in the partially quenched setting in figure 2. Let us briefly compare to how Σ and F_π were determined from lattice data in ref. [22]. In order to avoid two-parameter fits, there Σ was first determined at $\mu_{1,2} = 0$ from the first eigenvalue alone. Then, the two-point density correlation function $\rho(\hat{x}, \hat{y})$ of finding one \hat{x} - and one \hat{y} -eigenvalue was expanded for small $\hat{\delta}$, and F_π was fitted to the resulting Gaussian repulsion between \hat{x} and \hat{y} . We would expect that the detailed knowledge of the shape and spacing of several individual eigenvalues combined with independent measurements at different topologies could provide an alternative way to determine both low-energy constants.

5 Conclusions

We have derived new analytical expressions for the probability distribution of the k th lowest Dirac operator eigenvalue in QCD with three (or more) colours with spontaneous breaking of chiral symmetry. Extending earlier results we have shown how these distributions become modified when the Dirac operator couples to imaginary chemical potential, for arbitrary topology and including the partially quenched case. Because the deformation parameter of the spectrum is $\hat{\mu} = \mu F_\pi \sqrt{V}$ (where μ is

the externally supplied isospin chemical potential), one can use this to extract the magnitude of F_π through lattice gauge theory simulations. This has been already successfully done for the first eigenvalue at $\nu = 0$, after including first order finite-volume corrections via $F_{\pi eff}$. Our results thus offer further independent checks for higher topology or higher eigenvalues, given they remain in the ϵ -regime.

There is a clear sensitivity to F_π in these individual eigenvalue distributions, but it requires quark masses to be quite small. Because present-day lattice gauge configurations may tend to be available for relatively large masses, far from the scale of eigenvalues considered here, the eigenvalue distributions will look close to quenched and there is then little variation in the distributions as μ is introduced. For this reason it would be extremely helpful if one could derive analogous analytical expressions for mixed or conditional individual eigenvalue distributions, say to find simultaneously the first eigenvalue of \mathcal{D}_1 at x and the first eigenvalue of \mathcal{D}_2 at y . In the same way as the mixed spectral two-point density correlation function $\rho(x, y)$ considered previously in the literature, that develops a delta-function in the $\hat{\mu} \rightarrow 0$ limit, this quantity should lead to a more dramatic numerical signal as chemical potential is turned on and thus to an easier way of measuring F_π by means of such a technique. It is an open challenge to find new mathematical tricks that will be needed to perform this extension of our results.

ACKNOWLEDGEMENTS: We are indebted to Poul Damgaard for many discussions and an early collaboration on this topic. We would also like to thank Christoph Lehner and Tilo Wettig for discussions and correspondence regarding their work.

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A Identity for Laguerre Polynomials

In this Appendix we derive the form of the new polynomials given in eq. (3.17). They first appear in the manipulations leading to eq. (3.15) in the following form

$$\begin{aligned}
q_n^\nu(M_2^2) &\equiv \sum_{j=0}^n \frac{(-)^{n+j} n! (n+1)!}{(n-j)! j! (j+1)!} \sum_{l=0}^j \frac{\tau^l}{(1-\tau)^l} \hat{L}_l^\nu \left(\frac{M_2^2}{\tau} \right) (-X_k^2)^{j-l} \binom{j}{l} \\
&= (-)^n n! \sum_{l=0}^n \frac{\tau^l}{(1-\tau)^l} L_l^\nu \left(\frac{M_2^2}{\tau} \right) L_{n-l}^{l+1}(-X_k^2) \\
&= (-)^n n! \sum_{l=0}^n L_{n-l}^{l+1}(-X_k^2) \sum_{j=0}^l \frac{(-)^{j+l} (l+\nu)!}{(l-j)! (j+\nu)! (1-\tau)^j} L_j^\nu(M_2^2) \\
&= (-)^n n! \sum_{j=0}^n \frac{1}{(1-\tau)^j} L_j^\nu(M_2^2) \sum_{i=0}^{n-j} \frac{(-)^i (i+j+\nu)!}{(j+\nu)! i!} L_{n-j-i}^{i+j+1}(-X_k^2) . \tag{A.1}
\end{aligned}$$

The first line originates from generating polynomials \hat{L}_n^1 in the lower rows in eq. (3.15). Next we have swapped the sums, $\sum_{j=0}^n \sum_{l=0}^j \rightarrow \sum_{l=0}^n \sum_{j=l}^n$, so that the sum in $-X_k^2$ gives $L_{n-l}^{l+1}(-X_k^2)$. Then we use the following identity:

$$L_n^\nu(zw) = \sum_{j=0}^n \frac{(n+\nu)!}{(n-j)! (\nu+j)!} w^j (1-w)^{n-j} L_j^\nu(z) . \tag{A.2}$$

to eliminate the argument $1/\tau$ from the first Laguerre polynomial, choosing $z = M_2^2$ and $w = 1/\tau$. The newly appearing sum is swapped again to $\sum_{j=0}^n \sum_{l=j}^n$. The remaining sum can be simplified with

the help of the identity

$$L_m^{-\nu}(x) = \sum_{i=0}^m \frac{(-)^i (n + \nu - m + i)!}{(n + \nu - m)! i!} L_{m-i}^{n-m+i+1}(x) , \quad (\text{A.3})$$

with $m = n - j$ and $x = -X_k^2$.

The polynomials $q_n^\nu(x)$ and their derivation differ slightly from the polynomials $Q_n(x)$ in [13]. First, we allow for non-zero topology here, in contrast to there. Second, in the related formula to (3.15) the Laguerre polynomials L_n^0 have to be generated there, instead of L_n^1 here.

For completeness we also define the generalization of the polynomials $Q_n(x)$ that are needed to complete the computations of [13] for non-zero topology:

$$\begin{aligned} Q_n^\nu(M_2^2) &\equiv \sum_{j=0}^n \frac{(-)^{n+j} (n!)^2}{(n-j)! (j!)^2} \sum_{l=0}^j \frac{\tau^l}{(1-\tau)^l} \hat{L}_l^\nu \left(\frac{M_2^2}{\tau} \right) (-S^2)^{j-l} \binom{j}{l} \\ &= (-)^n n! \sum_{j=0}^n \frac{1}{(1-\tau)^j} L_j^\nu(M_2^2) L_{n-j}^{-\nu-1}(-S^2) , \end{aligned} \quad (\text{A.4})$$

with $S^2 = +N\tau c_1 s^2$. The derivation goes along exactly the same lines as above and we skip the details here. The single difference compared to eq. (3.17) is the shift in the negative index from $-\nu \rightarrow -\nu - 1$ in the final result. In Appendix C below we give the corresponding expressions for the first eigenvalue from [13] for non-zero ν .

B Integration Theorem over Determinants

In this Appendix we compute averages of the product of two determinants of different size that both contain different variables *and* different polynomials. The Theorem we give is a slight generalisation of Theorem 1 in [26]. It was already used in [13] but since the form we need here is more general we provide a short derivation. In particular, it generalises Dyson's Theorem, see Theorem 5.1.4 in [25], valid only for determinants of the same size and of the same polynomials.

Suppose we have two sets of (bi)orthonormal functions (this includes the case of polynomials orthogonal w.r.t. a given weight, since the weight can be absorbed into the functions) $\varphi_{j=0,1,\dots}(z)$ and $\tilde{\varphi}_{j=0,1,\dots}(z)$

$$\int dz \varphi_j(z) \tilde{\varphi}_l(z) = \delta_{jl} \text{ for } j, l = 0, 1, \dots . \quad (\text{B.1})$$

Furthermore we assume that we have two matrices (independent of z_1, \dots, z_N), A_{ij} and B_{ij} . We can then compute the following average over N variables z_j with $k \geq 1$:

$$\mathcal{I} \equiv \int_0^\infty \prod_{j=1}^N dz_j \left| \begin{array}{ccc} A_{1,0} & \cdots & A_{1,N+k-1} \\ \vdots & & \vdots \\ A_{k,0} & \cdots & A_{k,N+k-1} \\ \varphi_0(z_1) & \cdots & \varphi_{N+k-1}(z_1) \\ \vdots & & \vdots \\ \varphi_0(z_N) & \cdots & \varphi_{N+k-1}(z_N) \end{array} \right| \left| \begin{array}{ccc} B_{1,0} & \cdots & B_{1,N+k'-1} \\ \vdots & & \vdots \\ B_{k',0} & \cdots & B_{k',N+k'-1} \\ \tilde{\varphi}_0(z_1) & \cdots & \tilde{\varphi}_{N+k'-1}(z_1) \\ \vdots & & \vdots \\ \tilde{\varphi}_0(z_N) & \cdots & \tilde{\varphi}_{N+k'-1}(z_N) \end{array} \right| , \quad (\text{B.2})$$

where without loss of generality we assume $k \geq k'$. We need to define 4 different objects that we refer to as kernels

$$K_{N+k'}^I(i, j) = \sum_{l=0}^{N+k'-1} A_{i,l} B_{j,l} , \quad K_{N+k'}^{II}(i, z) = \sum_{l=0}^{N+k'-1} A_{i,l} \tilde{\varphi}_l(z) ,$$

$$K_{N+k'}^{III}(z, j) = \sum_{l=0}^{N+k'-1} \varphi_l(z) B_{j,l} , \quad K_{N+k'}^{IV}(z, z') = \sum_{l=0}^{N+k'-1} \varphi_l(z) \tilde{\varphi}_l(z') . \quad (\text{B.3})$$

Only the last kernel has the self-reproducing properties required for Dyson's Theorem to apply:

$$\begin{aligned} \int dz' K_{N+k'}^{IV}(z, z') K_{N+k'}^{IV}(z', z'') &= K_{N+k'}^{IV}(z, z'') , \\ \int dz' w(z') K_{N+k'}^{IV}(z', z') &= N + k' . \end{aligned} \quad (\text{B.4})$$

Obviously the contraction of mixed kernels leads to mixed kernels,

$$\begin{aligned} \int dz' K_{N+k'}^{II}(i, z') K_{N+k'}^{III}(z', j) &= K_{N+k'}^I(i, j) , \\ \int dz' K_{N+k'}^{II}(i, z') K_{N+k'}^{IV}(z', z) &= K_{N+k'}^{II}(i, z) , \\ \int dz' K_{N+k'}^{IV}(z, z') K_{N+k'}^{III}(z', i) &= K_{N+k'}^{III}(z, i) . \end{aligned} \quad (\text{B.5})$$

The result is now given in terms of the matrix

$$\mathcal{B}_{ij} = \begin{cases} K_{N+k'}^I(i, j) & i = 1, \dots, k, j = 1, \dots, k' \\ A_{i, j+N-1} & i = 1, \dots, k, j = k' + 1, k' + 2, \dots, k \end{cases} .$$

The claim is that the integral (B.2) is given by

$$\mathcal{I} = (-)^{N(k-k')} N! \det[\mathcal{B}] . \quad (\text{B.6})$$

By choosing A_{ij} and B_{ij} to be appropriate polynomials of the unintegrated variables, letting $\varphi_n(z)$ and $\tilde{\varphi}_n(z)$ be Laguerre polynomials times the weight factor, and choosing $N - k$ variables $z_1, \dots, z_N \rightarrow Z_{k+1}, \dots, Z_N$ we arrive at the integral of (3.19). Taking the integration norm into account, we see that (3.23), (3.24) and (3.25) follow from (B.6).

The proof of the formula is by induction on k . We first consider the base case $k = k'$, where (B.6) is just $N! \det[K^I(i, j)]$. To derive this, we first transpose one of the matrices in (B.2), and then multiply them inside a common determinant. The resulting matrix will then consist of blocks with each of the four types of kernels. Using the self-reproducing property of K^{VI} and Dyson's Theorem [25], we can carry out the integration, yielding the stated result.

For the step $k \rightarrow k + 1$, we expand the first determinant of (B.2) in the last column:

$$\sum_{i=1}^{k+N+1} (-)^{k+N+1+i} \int_0^\infty \left(\prod_{j=1}^N dz_j \right) (M_A)_{i, k+N+1} \det[m_A(i)] \det[M_B],$$

with M_A (M_B) the first (second) matrix of (B.2), and $m_A(i)$ being M_A with the row i and column $k + N + 1$ removed. The integrands with $i > k + 1$ are proportional to $\varphi_{k+N}(z_{i-k-1})$, but since there is no corresponding $\tilde{\varphi}_{k+N}(z_{i-k-1})$ factor in the expansion of the M_B determinant, these terms are killed by orthogonality. Using the induction hypothesis on the remaining terms (proportional to $A_{i, k+N}$) and comparing to the expansion of $\det \mathcal{B}$ in the last column, we verify (B.6).

C Non-Zero Topology for $p_{k=1}^{(N_f, \nu)}(x)$: an Alternative Formulation

In this appendix we derive an alternative formulation of the distribution of the first eigenvalue for $\nu > 0$. It is based on an earlier paper [13] to where we refer for more details, explicit results were given there only for $\nu = 0$. Because the calculations in the main part of this paper are quite involved it is useful to have an independent, equivalent result as a cross check.

In [13] one computes first the gap probability⁴, that the interval $[0, s]$ is empty of all eigenvalues x_i :

$$E_\nu^{(N_f)}(s) \equiv \frac{1}{\mathcal{Z}_\nu^{(N_f)}} \int_s^\infty dx_1 \dots dx_N \int_0^\infty dy_1 \dots dy_N \mathcal{P}_\nu^{(N_f)}(\{x\}, \{y\}; \{m_1\}, \{m_2\}) . \quad (\text{C.1})$$

From this the distribution of the first eigenvalue follows by differentiation, $p_1^{(N_f, \nu)}(s) = -\partial_s E_\nu^{(N_f)}(s)$. In steps very similar to the main body of this paper we obtain

$$E_\nu^{(N_f)}(s) \sim \frac{1}{\mathcal{Z}_\nu^{(N_1+N_2)} \Delta_{N_1+\nu}(m_{f_1}'^2) \Delta_{N_2}(m_{f_2}^2)} e^{-N^2 \tau c_1 s^2} \int_0^\infty dz_1 \dots dz_N e^{-N \sum_{i=1}^N \tau c_1 z_i} \quad (\text{C.2})$$

$$\times \begin{vmatrix} \hat{L}_0(M_1'^2) & \dots & \hat{L}_{N+N_1+\nu-1}(M_1'^2) \\ \dots & \dots & \dots \\ \hat{L}_0(M_{N_1+\nu}'^2) & \dots & \hat{L}_{N+N_1+\nu-1}(M_{N_1+\nu}'^2) \\ \hat{L}_0(Z_1) & \dots & \hat{L}_{N+N_1+\nu-1}(Z_1) \\ \dots & \dots & \dots \\ \hat{L}_0(Z_N) & \dots & \hat{L}_{N+N_1+\nu-1}(Z_N) \end{vmatrix} \begin{vmatrix} Q_0^\nu(M_1^2) & \dots & Q_{N+N_2-1}^\nu(M_1^2) \\ \dots & \dots & \dots \\ Q_0^\nu(M_{N_2}^2) & \dots & Q_{N+N_2-1}^\nu(M_{N_2}^2) \\ \hat{L}_0(Z_1) & \dots & \hat{L}_{N+N_2-1}(Z_1) \\ \dots & \dots & \dots \\ \hat{L}_0(Z_N) & \dots & \hat{L}_{N+N_2-1}(Z_N) \end{vmatrix} ,$$

where we have suppressed all mass independent normalisation factors. The Laguerre polynomials $\hat{L}_k(x) = (-)^k k! L_k(x)$ in monic normalisation contain the following masses:

$$M_j^2 \equiv -N \tau c_2 m_j^2, \quad j = 1, \dots, N_2 \quad (\text{C.3})$$

for the flavours of type N_2 , and for flavour N_1 we have ν additional masses

$$\begin{aligned} M_j'^2 &\equiv -N \tau c_1 m_j'^2 \equiv -N \tau c_1 (m_j^2 + s^2), \quad j = 1, \dots, N_1, \\ M_j'^2 &\equiv -N \tau c_1 m_j'^2 \equiv -N \tau c_1 (s^2 + \epsilon_j), \quad j = N_1 + 1, \dots, N_1 + \nu. \end{aligned} \quad (\text{C.4})$$

The degeneracy of the latter can be reinstated by setting $\epsilon_j = 0$ at the end of the calculation. Alternatively we could write out the resulting derivatives acting on the first determinant in eq. (C.2) for $\nu > 1$. Last but not least we have introduced a set of new polynomials containing now generalised Laguerre polynomials, as defined in (A.4). This completes in principle the computations of [13] for non-zero topology. The derivation from the first to the last line in eq. (A.4) goes along the same lines as the previous appendix A for the polynomials $q_n^\nu(x)$, apart from the slightly different identity to be used:

$$L_{n-j}^{-\nu-1}(x) = \sum_{i=0}^{n-j} \frac{(-)^i (j + \nu + i)!}{(j + \nu)! i!} L_{n-j-i}^{j+i}(x) . \quad (\text{C.5})$$

Applying the Theorem derived in the next appendix D the N integrals over variables z_k can now be easily performed. Instead of giving the most general result, distinguishing between different cases depending on the numbers N_1 , N_2 and ν , we give a few simple examples for illustration. These can be stated in terms of the above polynomials as well as the following kernel:

$$\mathcal{K}_N^\nu(M_1'^2, M_2^2) \equiv \sum_{j=0}^{N-1} \frac{(-)^j}{j!} L_j^0(M_1'^2) Q_j^\nu(M_2^2) . \quad (\text{C.6})$$

⁴It is denoted there by $E_{0,0}(s, t = 0)$.

Note that only the second polynomial Q_j^ν gets modified when $\nu \neq 0$, compared to [13]. The kernel is obviously not symmetric in its arguments.

In the first example we choose $N_1 = N_2 = 1$ to get

$$\begin{aligned} E_{\nu=0}^{(1+1)}(s) &\sim \frac{e^{-N^2 \tau c_1 s^2}}{\mathcal{Z}_0^{(1+1)}(m_1; m_2)} \mathcal{K}_{N+1}^{\nu=0}(M_1'^2, M_2'^2) , \\ E_{\nu=1}^{(1+1)}(s) &\sim \frac{e^{-N^2 \tau c_1 s^2}}{\mathcal{Z}_1^{(1+1)}(m_1; m_2)(m_1'^2 - s^2)} \begin{vmatrix} \mathcal{K}_{N+1}^{\nu=1}(M_1'^2, M_2'^2) & \mathcal{K}_{N+1}^{\nu=1}(S^2, M_2'^2) \\ L_{N+1}^0(M_1'^2) & L_{N+1}^0(S^2) \end{vmatrix} , \end{aligned} \quad (\text{C.7})$$

etc., with more rows with Laguerre polynomials for higher ν .

The second example which is probably most relevant for applications is partially quenched, with $N_1 = 0$ and $N_2 = 2$. Here the size of the determinant does not grow immediately as the masses $m_{1,2}$ of flavour N_2 get paired with those generated by topology, $\nu = 1, 2$ in our examples:

$$\begin{aligned} E_{\nu=0}^{(0+2)}(s) &\sim \frac{e^{-N^2 \tau c_1 s^2}}{\mathcal{Z}_0^{(0+2)}(m_{1,2})(m_1^2 - m_2^2)} \begin{vmatrix} Q_N^{\nu=0}(M_1^2) & Q_N^{\nu=0}(M_2^2) \\ Q_{N+1}^{\nu=0}(M_1^2) & Q_{N+1}^{\nu=0}(M_2^2) \end{vmatrix} , \\ E_{\nu=1}^{(0+2)}(s) &\sim \frac{e^{-N^2 \tau c_1 s^2}}{\mathcal{Z}_1^{(0+2)}(m_{1,2})(m_1^2 - m_2^2)} \begin{vmatrix} \mathcal{K}_{N+1}^{\nu=1}(S^2, M_1^2) & \mathcal{K}_{N+1}^{\nu=1}(S^2, M_2^2) \\ Q_{N+1}^{\nu=1}(M_1^2) & Q_{N+1}^{\nu=1}(M_2^2) \end{vmatrix} , \\ E_{\nu=2}^{(0+2)}(s) &\sim \frac{e^{-N^2 \tau c_1 s^2}}{\mathcal{Z}_2^{(0+2)}(m_{1,2})(m_1^2 - m_2^2)2s} \begin{vmatrix} \mathcal{K}_{N+1}^{\nu=2}(S^2, M_1^2) & \mathcal{K}_{N+1}^{\nu=2}(S^2, M_2^2) \\ \partial_s \mathcal{K}_{N+1}^{\nu=2}(S^2, M_1^2) & \partial_s \mathcal{K}_{N+1}^{\nu=2}(S^2, M_2^2) \end{vmatrix} . \end{aligned} \quad (\text{C.8})$$

In all cases one further differentiation with respect to s yields the distribution of the first eigenvalue.

C.1 The large- N limit

We will sketch here first how to take the microscopic large- N limit of the main building blocks: the generalised Laguerre polynomials $L_j^{\pm\alpha}$, the new polynomials Q_j^ν and the kernel \mathcal{K}_N^ν . Then we will give two explicit examples for the distribution of the first eigenvalue at $\nu > 0$.

The scaling limit is exactly as described in the main body of this paper (and in [13]), so we can be brief. We begin with the new polynomials Q_j^ν . The sum can be replaced by an integral except for a few terms, and we need the following ingredients in the limit $N \rightarrow \infty$ together with $\mu_{1,2} \rightarrow 0$

$$\begin{aligned} \lim_{N,j,k \rightarrow \infty} (1 - \tau)^{-k} &= \exp \left[\frac{1}{2} r t \hat{\delta}^2 \right] \quad \text{where } t \equiv j/N, \quad r \equiv k/j , \\ \lim_{N \rightarrow \infty} \tau c_{1,2} &= 1 . \end{aligned} \quad (\text{C.9})$$

For the Laguerre polynomials the following scaling holds:

$$\lim_{N,j \rightarrow \infty} L_j^\nu(M^2) = N^\nu (rt)^{\nu/2} (2/\hat{m})^\nu I_\nu(\hat{m}\sqrt{rt}) , \quad (\text{C.10})$$

$$\lim_{N,j,k \rightarrow \infty} L_{j-k>\nu}^{-\nu-1}(-S^2) = N^{-\nu-1} (t(1-r))^{-(\nu+1)/2} (\hat{s}/2)^{\nu+1} I_{\nu+1}(\hat{s}\sqrt{1-r}) . \quad (\text{C.11})$$

In comparison to $\nu = 0$, where we had to split off the s -independent part $L_0^{-1} = 1$ from the sum in eq. (A.4), we now have to separate a total number of $\nu + 1$ terms $L_{0,1,\dots,\nu}^{-\nu-1}$ from the sum. In the large- N limit these terms will seem to be of higher order, but after some cancelations taking place they will give a contribution of the same order as the sum itself.

To illustrate this we give the first two examples (including the known $\nu = 0$ case) for the limiting new polynomials $\lim_{j \rightarrow \infty} \frac{(-)^j}{j!} Q_j^\nu(M^2) \equiv Q_S^\nu(\hat{m}; t)$:

$$\begin{aligned}
Q_S^{\nu=0}(\hat{m}; t) &= \frac{\hat{s}}{2} \int_0^1 dr \frac{\sqrt{t} e^{\frac{1}{2}rt\hat{\delta}^2}}{\sqrt{1-r}} I_0(\hat{m}\sqrt{rt}) I_1(\hat{s}\sqrt{(1-r)t}) + e^{\frac{1}{2}t\hat{\delta}^2} I_0(\sqrt{t} \hat{m}) , \\
Q_S^{\nu=1}(\hat{m}; t) &= \frac{\hat{s}^2}{2\hat{m}} \int_0^1 dr \frac{\sqrt{rt} e^{\frac{rt}{2}\hat{\delta}^2}}{1-r} I_1(\hat{m}\sqrt{rt}) I_2(\hat{s}\sqrt{t(1-r)}) \\
&\quad + e^{\frac{t}{2}\hat{\delta}^2} \left[I_0(\hat{m}\sqrt{t}) + \frac{\sqrt{t} I_1(\hat{m}\sqrt{t})}{\hat{m}} \left(\hat{\delta}^2 + \frac{\hat{s}^2}{2} \right) \right] , \\
Q_S^{\nu=2}(\hat{m}; t) &= \frac{\hat{s}^3}{2\hat{m}^2} \int_0^1 dr \frac{r\sqrt{t} e^{\frac{rt}{2}\hat{\delta}^2}}{(1-r)^{\frac{3}{2}}} I_2(\hat{m}\sqrt{rt}) I_3(\hat{s}\sqrt{t(1-r)}) \\
&\quad + e^{\frac{t}{2}\hat{\delta}^2} \left[I_0(\hat{m}\sqrt{t}) + \frac{\sqrt{t}}{\hat{m}} I_1(\hat{m}\sqrt{t}) \left(\frac{\hat{s}^2}{2} + 2\hat{\delta}^2 \right) + \frac{t}{\hat{m}^2} I_2(\hat{m}\sqrt{t}) \left(\frac{\hat{s}^4}{8} + \frac{\hat{s}^2\hat{\delta}^2}{2} + \hat{\delta}^4 \right) \right] .
\end{aligned} \tag{C.12}$$

The fact that we get the following relation $\lim_{\mu_{1,2} \rightarrow 0} (-)^n Q_n^\nu(M_2^2)/n! = L_n^0(-N(m_2^2 + s^2))$ from eq. (A.4) in the limit of vanishing chemical potentials, implies a series of generalised Sonine identities, e.g. for $\nu = 1$:

$$\frac{s^2}{2m} \left(\int_0^1 dr \frac{\sqrt{r}}{1-r} I_1(m\sqrt{r}) I_2(s\sqrt{1-r}) + I_1(m) \right) + I_0(m) = I_0(\sqrt{m^2 + s^2}) , \tag{C.13}$$

etc. The general identity obtained in this way is stated in the following appendix and proven by induction.

The asymptotic limit of Q_S^ν then easily translates into the limiting form of the microscopic kernel:

$$\lim_{N \rightarrow \infty} \frac{1}{N} \mathcal{K}_N^\nu(M_1^2, M_2^2) \equiv \mathcal{K}_S^\nu(\hat{m}_1, \hat{m}_2) = \frac{1}{2} \int_0^1 dt I_0(\hat{m}_1 \sqrt{t}) Q_S^\nu(\hat{m}_2; t) . \tag{C.14}$$

We are now ready to give some explicit examples for gap probabilities at non-zero topology. The first example is for one flavour, the simplest partially quenched case, with $N_1 = 0$ and $N_2 = 1$ with mass \hat{m} . While for $\nu = 0$ we had [13]

$$E_{S,\nu=0}^{(0+1)}(\hat{s}) = \exp \left[-\frac{1}{4}\hat{s}^2 - \frac{1}{2}\hat{\delta}^2 \right] \frac{Q_S^{\nu=0}(\hat{m}; t=1)}{I_0(\hat{m})} , \tag{C.15}$$

our new results for topology $\nu = 1, 2$ are

$$E_{S,\nu=1}^{(0+1)}(\hat{s}) = \exp \left[-\frac{1}{4}\hat{s}^2 - \frac{1}{2}\hat{\delta}^2 \right] \frac{\hat{m}}{I_1(\hat{m})} \mathcal{K}_S^{\nu=1}(\hat{s}, \hat{m}) , \tag{C.16}$$

$$E_{S,\nu=2}^{(0+1)}(\hat{s}) = \exp \left[-\frac{1}{4}\hat{s}^2 - \frac{1}{2}\hat{\delta}^2 \right] \frac{\hat{m}^2}{\hat{s} I_2(\hat{m})} \left| \begin{array}{cc} \mathcal{K}_S^{\nu=2}(\hat{s}, \hat{m}) & \partial_p \mathcal{K}_S^{\nu=2}(p, \hat{m})|_{p=\hat{s}} \\ I_0(\hat{s}) & I_1(\hat{s}) \end{array} \right| . \tag{C.17}$$

Note that the normalising partition function is chosen such that it does not vanish at zero mass, e.g. $\mathcal{Z}_\nu^{(0+1)}(m) = I_\nu(m)/m^\nu$. It also gives rise to an extra factor $e^{\hat{\delta}^2}$ per unpaired flavour. The corresponding first eigenvalues are shown in figure 4 together with the corresponding densities from [12]:

$$\rho_\nu^{(0+1)}(\hat{x}) = \rho_\nu^{quen}(\hat{x}) - \exp \left[-\frac{1}{2}\hat{\delta}^2 \right] \hat{x} \frac{J_\nu(\hat{x})}{I_\nu(\hat{m}_1)} \int_0^1 dT T e^{\frac{1}{2}T^2\hat{\delta}^2} I_\nu(T\hat{m}_1) J_\nu(T\hat{x}) , \tag{C.18}$$

where the quenched one-matrix model spectral density reads [2]

$$\rho_\nu^{quen}(\hat{x}) \equiv \frac{\hat{x}}{2} (J_\nu(\hat{x})^2 - J_{\nu-1}(\hat{x})J_{\nu+1}(\hat{x})) . \quad (C.19)$$

It can be seen that our new results for individual eigenvalue distributions nicely follow the corre-

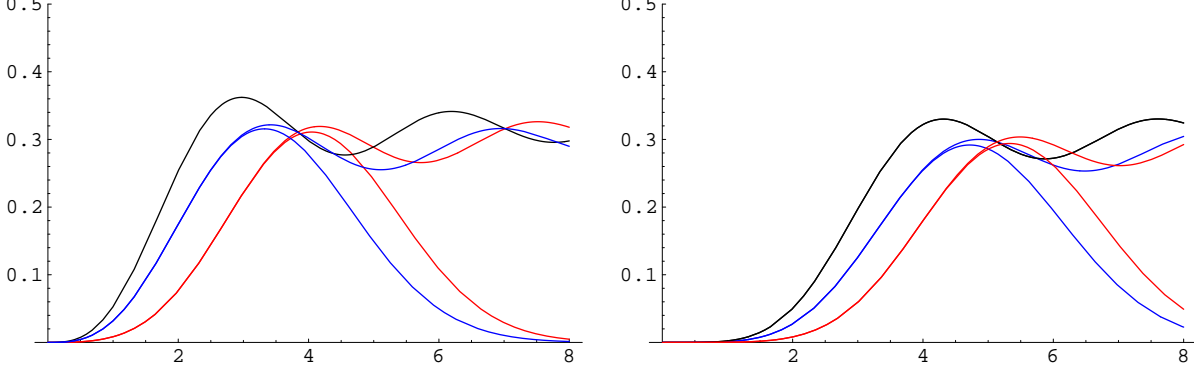


Figure 4: The partially quenched case with $N_1 = 0$ and $N_2 = 1$, at topology $\nu = 1$ (left plot) and $\nu = 2$ (right plot). The first eigenvalue is compared with the density eq. (C.18), for $\hat{m} = 1.5$ at $\hat{\delta} = 2.5$ (middle blue curves) and for $\hat{m} = 1.5$ at $\hat{\delta} = 0.5$ (right red curves). The latter is almost indistinguishable from the $N_f = 1$ flavour one-matrix model result [14, 15] (not shown), to which the two-matrix model reduces at $\hat{\delta} \rightarrow 0$. We also display the quenched one-matrix model density from eq. (C.19) at the corresponding value of $\nu = 1$ and 2 for comparison (left black curves). This curve is approached both when $\hat{m} \gg 1$ or $\hat{\delta} \gg 1$.

sponding known spectral densities almost up to the first local maximum, for the two different values of topology and various values of \hat{m} and $\hat{\delta}$ chosen.

The second set of examples is the gap for two flavours. First we give the case $N_1 = 1 = N_2$ at topology $\nu = 1$ (and 0):

$$E_{S,\nu=0}^{(1+1)}(\hat{s}) = \frac{e^{-\hat{s}^2/4}}{\mathcal{Z}_{\nu=0}^{(1+1)}(\hat{m}_1; \hat{m}_2)} \mathcal{K}_S^{\nu=0}(\hat{m}'_1, \hat{m}_2) ,$$

$$E_{S,\nu=1}^{(1+1)}(\hat{s}) = \frac{e^{-\hat{s}^2/4}}{\mathcal{Z}_{\nu=1}^{(1+1)}(\hat{m}_1; \hat{m}_2)(\hat{m}_1'^2 - \hat{s}^2)} \begin{vmatrix} \mathcal{K}_S^{\nu=1}(\hat{m}'_1, \hat{m}_2) & \mathcal{K}_S^{\nu=1}(\hat{s}, \hat{m}_2) \\ I_0(\hat{m}'_1) & I_0(\hat{s}) \end{vmatrix} , \quad (C.20)$$

$$\text{with } \mathcal{Z}_{\nu}^{(1+1)}(\hat{m}_1; \hat{m}_2) = \frac{1}{2(\hat{m}_1 \hat{m}_2)^\nu} \int_0^1 dt e^{\hat{\delta}^2 t/2} I_\nu(\hat{m}_1 \sqrt{t}) I_\nu(\hat{m}_2 \sqrt{t}) . \quad (C.21)$$

The corresponding first eigenvalue distributions obtained by differentiating these quantities with respect to s are plotted in figure 5 against the corresponding spectral density [12],

$$\rho_\nu^{(1+1)}(\hat{x}) = \rho_\nu^{quen}(\hat{x}) - \hat{x} \frac{\int_0^1 dt t J_\nu(t\hat{x}) I_\nu(t\hat{m}_1) \int_0^1 dt t e^{\frac{1}{2}t^2 \hat{\delta}^2} J_\nu(t\hat{x}) I_\nu(t\hat{m}_2)}{\int_0^1 dt t e^{\frac{1}{2}t^2 \hat{\delta}^2} I_\nu(t\hat{m}_1) I_\nu(t\hat{m}_2)} . \quad (C.22)$$

Here the difference between 1 + 1 flavours (left) and partial quenching (right) becomes visible. Once more the individual eigenvalue distributions agree nicely with the corresponding spectral densities.

The partially quenched two-flavour case, with $N_1 = 0$ and $N_2 = 2$ reads as follows in the large- N limit:

$$E_{\nu=1}^{(0+2)}(s) = \frac{e^{-\hat{s}^2/4 - \hat{\delta}^2} \hat{m}_1 \hat{m}_2}{\hat{m}_1 I_2(\hat{m}_1) I_1(\hat{m}_2) - \hat{m}_2 I_2(\hat{m}_2) I_1(\hat{m}_1)} \begin{vmatrix} \mathcal{K}_S^{\nu=1}(\hat{s}, \hat{m}_1) & \mathcal{K}_S^{\nu=1}(\hat{s}, \hat{m}_2) \\ Q_S^{\nu=1}(\hat{m}_1; t=1) & Q_S^{\nu=1}(\hat{m}_2; t=1) \end{vmatrix} . \quad (C.23)$$

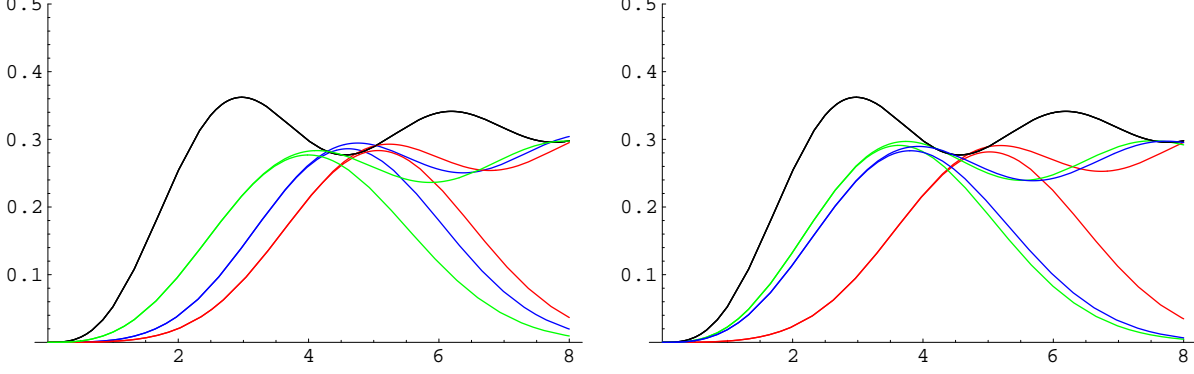


Figure 5: The example of two flavours: $N_1 = N_2 = 1$ (left plot), and the partially quenched case with $N_1 = 0$ and $N_2 = 2$ (right plot), both at topology $\nu = 1$. The first eigenvalue is compared with the corresponding two-matrix model densities, for the same values of parameters: $\hat{m}_1 = 1.5$, $\hat{m}_2 = 3.5$ at $\hat{\delta} = 0.5$ (right red curves), the same masses at $\hat{\delta} = 2.5$ (middle blue curves), and $\hat{m}_1 = 5.5$, $\hat{m}_2 = 3.5$ at $\hat{\delta} = 2.5$ (left green curves). The quenched one-matrix model density from eq. (C.19) at $\nu = 1$ (top black curves) is shown to guide the eye. The curves at smallest $\hat{\delta} = 0.5$ (right red curves) almost agree for both flavour settings, being very close to the the $N_f = 2$ flavour one-matrix model result [14] (not shown). However, the other two curves differentiate between $N_f = 1 + 1$ and $N_f = 0 + 2$ flavours for the same parameter values.

Note again the proper normalisation of the partition function in masses and $\hat{\delta}$. The corresponding density to compare with was derived in [12] where it is given as a determinant (including the corrections in the Erratum)

$$\begin{aligned} \rho_\nu^{(N_f=0+2)}(\hat{x}) &= \rho_\nu^{quen}(\hat{x}) - \exp\left[-\frac{1}{2}\hat{\delta}^2\right] \hat{x} \left(\hat{m}_1 I_{\nu+1}(\hat{m}_1) I_\nu(\hat{m}_2) - \hat{m}_2 I_\nu(\hat{m}_1) I_{\nu+1}(\hat{m}_2)\right)^{-1} \\ &\times \left[\int_0^1 dt t e^{\frac{1}{2}\hat{\delta}^2 t^2} J_\nu(\hat{x}t) I_\nu(\hat{m}_1 t) \left(-I_\nu(\hat{m}_2)(\hat{x} J_{\nu+1}(\hat{x}) + \hat{\delta}^2 J_\nu(\hat{x})) - \hat{m}_2 I_{\nu+1}(\hat{m}_2) J_\nu(\hat{x}) \right) \right. \\ &\quad \left. + \int_0^1 dt t e^{\frac{1}{2}\hat{\delta}^2 t^2} J_\nu(\hat{x}t) I_\nu(\hat{m}_2 t) \left(I_\nu(\hat{m}_1)(\hat{x} J_{\nu+1}(\hat{x}) + \hat{\delta}^2 J_\nu(\hat{x})) + \hat{m}_1 I_{\nu+1}(\hat{m}_1) J_\nu(\hat{x}) \right) \right], \end{aligned} \quad (C.24)$$

In all cases where we could compare to the approach presented in the main body of this paper the curves obtained from the two equivalent approaches agree.

D Sonine Identities

In this appendix we prove some integral identities relevant for the appendix C that have to be satisfied and that can be reduced to the so-called Sonine integral eq. (D.1) below.

We mentioned already that as a check in the limit of vanishing chemical potential the new polynomials Q_n^ν have to reduce to the Laguerre polynomials at shifted mass, in order to reproduce the known one-matrix models results. While for finite- N this can be done using the first line of the definition eq. (A.4), this is not so easy after taking the large- N limit. In fact from eq. (C.12) at $\hat{\delta} = 0$ and $t = 1$ ⁵

⁵The case $t \neq 1$ is easily reestablished when rescaling $\hat{m}, \hat{s} \rightarrow \sqrt{t}\hat{m}, \sqrt{t}\hat{s}$.

we get the following identities for $\nu = 0, 1, 2$:

$$\mathcal{I}_0 \equiv s \int_0^1 dx \frac{x}{\sqrt{1-x^2}} I_0(mx) I_1(s\sqrt{1-x^2}) + I_0(m) - I_0(\sqrt{m^2+s^2}) = 0, \quad (\text{D.1})$$

$$\mathcal{I}_1 \equiv \frac{s^2}{m} \int_0^1 dx \frac{x^2}{1-x^2} I_1(mx) I_2(s\sqrt{1-x^2}) + \frac{s^2}{2m} I_1(m) + I_0(m) - I_0(\sqrt{m^2+s^2}) = 0, \quad (\text{D.2})$$

$$\begin{aligned} \mathcal{I}_2 &\equiv \frac{s^3}{m^2} \int_0^1 dx \frac{x^3}{(1-x^2)^{\frac{3}{2}}} I_2(mx) I_3(s\sqrt{1-x^2}) + \frac{s^4}{8m^2} I_2(m) + \frac{s^2}{2m} I_1(m) + I_0(m) - I_0(\sqrt{m^2+s^2}) \\ &= 0. \end{aligned} \quad (\text{D.3})$$

The first equation is the known Sonine identity, see e.g. [27], [28] as well as [13] for an independent derivation. It is easier to state and prove the difference between two consecutive integral identities of this kind:

$$\begin{aligned} 0 = \mathcal{I}_\nu - \mathcal{I}_{\nu-1} &= \frac{s^{\nu+1}}{m^\nu} \int_0^1 dx \frac{x^{\nu+1}}{(1-x^2)^{\frac{\nu+1}{2}}} I_\nu(mx) I_{\nu+1}(s\sqrt{1-x^2}) + \frac{s^{2\nu}}{2^\nu \nu! m^\nu} I_\nu(m) - \\ &\quad - \frac{s^\nu}{m^{\nu-1}} \int_0^1 dx \frac{x^\nu}{(1-x^2)^{\frac{\nu}{2}}} I_{\nu-1}(mx) I_\nu(s\sqrt{1-x^2}). \end{aligned} \quad (\text{D.4})$$

Because the way to prove the induction start $\mathcal{I}_1 - \mathcal{I}_0 = 0$ and the induction step is the same, using integration by parts, we will be brief. If we use

$$\partial_y \left(\frac{I_\nu(y)}{y^\nu} \right) = \frac{I_{\nu+1}(y)}{y^\nu}, \quad (\text{D.5})$$

upon choosing $y = s\sqrt{1-x^2}$ we can write

$$\frac{x I_{\nu+1}(s\sqrt{1-x^2})}{(1-x^2)^{(\nu+1)/2}} = -\partial_x \left(\frac{I_\nu(s\sqrt{1-x^2})}{s(1-x^2)^{\nu/2}} \right). \quad (\text{D.6})$$

We can therefore rewrite the first integral in eq. (D.4) as follows:

$$\begin{aligned} &\int_0^1 dx \frac{x^{\nu+1}}{(1-x^2)^{\frac{\nu+1}{2}}} I_\nu(mx) I_{\nu+1}(s\sqrt{1-x^2}) = \\ &= - \left. \frac{x^\nu I_\nu(mx) I_\nu(s\sqrt{1-x^2})}{s(1-x^2)^{\nu/2}} \right|_0^1 + \int_0^1 dx \frac{I_\nu(s\sqrt{1-x^2})}{s(1-x^2)^{\nu/2}} x^\nu m I_{\nu-1}(mx) \\ &= - \frac{I_\nu(m)}{s} \left(\frac{s}{2} \right)^\nu \frac{1}{\nu!} + \frac{m}{s} \int_0^1 dx \frac{x^\nu}{(1-x^2)^{\frac{\nu}{2}}} I_{\nu-1}(mx) I_\nu(s\sqrt{1-x^2}), \end{aligned} \quad (\text{D.7})$$

where we have used a Bessel identity $p I_\nu(p)' + \nu I_\nu(p) = p I_{\nu-1}(p)$ as well as the series representation of the Bessel function to determine the limit at the upper bound $x = 1$. Inserting this for $\nu = 1$ and then general ν yields the induction start and induction step.

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